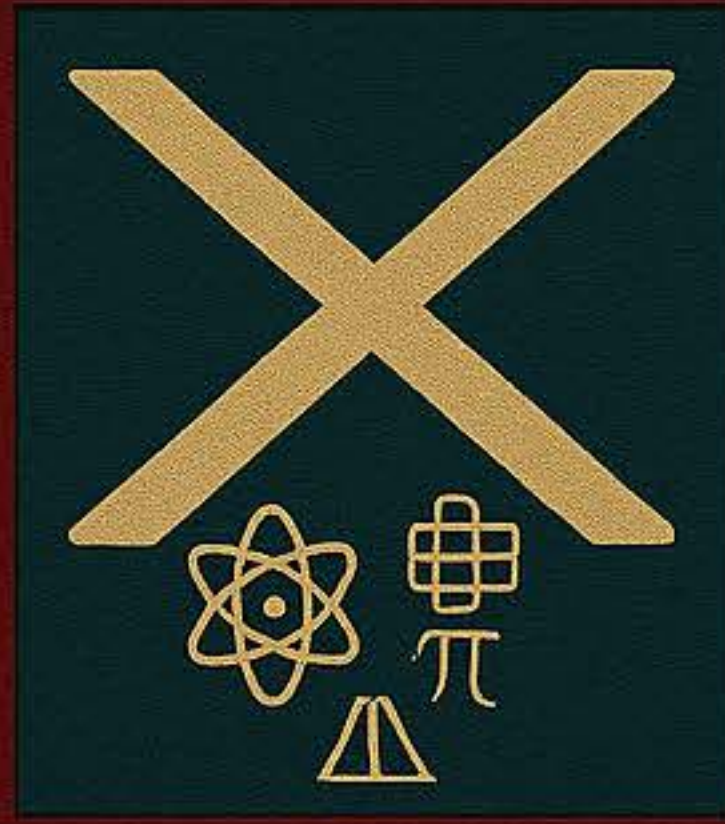




# PARTIAL DIFFERENTIAL EQUATIONS

CORE CONCEPTS AND ITS  
STOCHASTIC USES

Dr. Richard Murdoch  
Montgomery, MD, PhD

THE  
SCOTTISH  
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PARTIAL  
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MONTGOMERY

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# *Partial Differential Equations:*

## *Core Concepts and Its Stochastic Uses*

This volume presents a comprehensive and rigorous treatment of partial differential equations, from its foundations, progressing through elliptic, parabolic and hyperbolic equations. The book covers classification, techniques of solution, and stochastic applications. It culminates in the concept of regularity structures, deriving each fundamental result with meticulous mathematical precision.

Dr. Richard Murdoch Montgomery, holding both MD and PhD degrees, brings a unique interdisciplinary perspective to this authoritative exposition, combining deep theoretical insight with pedagogical clarity.

Published by The Scottish Science Society, this volume continues the tradition of rigorous academic scholarship in the mathematical sciences.

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THE SCOTTISH SCIENCE SOCIETY



# Partial Differential Equations: Core Concepts and Its Stochastic Uses

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*To my mother, last lighthouse when navigating into the darkest storms.*



## Prefácio

*“Evolução: uma teoria na História ou uma história sem fim?”*

Uma das teorias mais aceitas na contemporaneidade sobre o método científico é a dos paradigmas, popularizada por Thomas Kuhn. No entanto, esse modelo não dá conta de descrever como se dão as crises e quebras de paradigma ao longo da história, justamente porque é muito difícil para quem vive dentro de um determinado paradigma se aperceber de que o mesmo enfrenta graves desafios.

Em nosso ponto de vista, o reconhecimento do papel histórico e epistemológico das controvérsias pode apontar para a solução dos dilemas que envolvem os paradigmas em todas as épocas. Se olharmos para as grandes controvérsias do passado, constataremos que muitas delas não foram cabalmente dirimidas, ressurgindo na arena científica de tempos em tempos. Mais ainda, se considerarmos os grandes debates atuais como importantes não apenas quanto à definição de um lado “vencedor”, as discordâncias podem nos ensinar muito mais sobre como funciona a ciência do que as concordâncias.

Toda vez que chegamos ao limite momentâneo de uma ciência, verificamos que há domínios além dos conhecimentos daquela ciência e que se podem chamar de “metafísicos”. Com o passar do tempo esses domínios acabam sendo melhor entendidos e trazidos para dentro do escopo das ciências, mas é como se a barreira da metafísica se deslocasse sempre para um novo nível. Mais ainda: é a metafísica que alimenta muitas das fronteiras da ciência, colocando-lhe como desafio problemas ainda insolúveis dentro do estado da arte.

De fato, há uma série de problemas metafísicos não resolvidos pelas diversas ciências. Por exemplo, em física, não temos ideia de onde estão os limites da subdivisão da matéria e energia, pois hoje há teorias onde se fala em subquarks, quando até bem pouco tempo os quarks, que se candidatam a “blocos” constitutivos das chamadas partículas elementares, eram entidades completamente ignoradas. Um outro exemplo é o que ocorre quando se pergunta aos cientistas qual seria a estrutura de um fóton: essa busca pela essência parece atemorizá-los de tal forma que os leva a repelir a questão.

Não causa surpresa que a biologia esteja envolvida em uma série de grandes controvérsias que extravasam para a sociedade contemporânea, tais como: origem e essência da vida, existência de vida extraterrestre, processo da hominização, inteligência artificial, teleologia dos processos vitais, sociobiologia, manipulação genética ou impasses ecológicos.

Dentre os problemas da biologia, desde o século XIX tomou vulto o da evolução das espécies. As controvérsias que cercam a mais aceita das teorias da evolução, na sua roupagem atual do neodarwinismo, caem num terreno sensível, porque se tornou palco de ideologias que levam as pessoas a pensar que criticar essa teoria significa defender o criacionismo religioso na sua forma fundamentalista.



Para se entender como podem subsistir com base científica outras teorias evolutivas e conceitos como autocriação, ou que volta e meia se encontre a recorrência de questões metafísicas entre as teorias científicas, devemos nos perguntar quais foram ao longo da História as controvérsias dessas teorias e como foram enfrentadas.

Já em biologia não há consenso quanto à capacidade de se decifrar a formação da vida e de suas propriedades apenas pela decodificação dos genomas. Há na atualidade os que pensam ser cientificamente viável o caminho inverso da proteína para o DNA, rompendo a barreira de Weissmann, o que na prática significaria a possibilidade de herança de caracteres adquiridos na consequente aparição de novas espécies. Nesse contexto é interessante contrapor ao conceito de “relojoeiro cego” a ideia de um universo autocriador de sua própria ordem, imaginando como uma complexificação crescente levaria de moto próprio a evolução do inanimado para a vida e para o ponto de inflexão que foi o surgimento da humanidade.

Essas considerações preliminares se aplicam a Richard Montgomery, que é uma dessas raras pessoas que ultrapassam com felicidade os muros que a especialização positivista tem imposto como regra acadêmica e profissional. Sua inquietação intelectual o faz caminhar da neurologia para a economia, para as profundezas da inteligência artificial e de volta para a biologia e daí para a física, circulando pela reflexão matemática abstrata e aplicada e tergiversando alegremente com a História e a Filosofia. Essa disposição de criar um itinerário próprio vale a pena ser conhecida e discutida.

Gildo Magalhães  
Professor Titular  
Centro de História da Ciência  
Universidade de São Paulo



# General Introduction

## Introduction

This habilitation thesis presents a body of work situated at the intersection of stochastic analysis, partial differential equations, and mathematical physics. The overarching objective is to develop and apply advanced mathematical tools to the study of complex systems evolving under the influence of random fluctuations. The research presented herein progresses from foundational concepts in stochastic calculus and the theory of well-posed stochastic partial differential equations (SPDEs) to the frontier of the field: the analysis of singular SPDEs, which have long posed a formidable challenge to the mathematical community.

The classical theory of SPDEs, built upon the foundational work of Itô and others, provides a robust framework for a wide class of equations where the nonlinearities are sufficiently regular with respect to the driving noise. The early chapters of this thesis review and extend this classical framework, exploring [mention topics from early chapters, e.g., the application of Malliavin calculus, the study of large deviation principles for certain classes of SPDEs, or the analysis of specific models from fluid dynamics or finance]. These investigations establish the groundwork and highlight the limitations of classical methods when confronted with the highly irregular solutions and ill-posed nonlinearities characteristic of singular SPDEs.

Such singular equations are not mere mathematical curiosities; they are central to the description of fundamental physical phenomena. Models such as the Kardar-Parisi-Zhang (KPZ) equation, which describes the universal behaviour of growing interfaces, and the dynamic  $\Phi_3^4$  model from quantum field theory, fall into this category. For decades, a rigorous mathematical understanding of these equations remained elusive, with progress being driven primarily by the non-rigorous but powerfully predictive methods of theoretical physics, most notably the renormalisation group.

The central contribution of this thesis, presented in the final chapter, is an original and self-contained exposition of the modern mathematical theory that has finally provided a rigorous foundation for the study of these singular SPDEs. This work culminates in a detailed presentation of Martin Hairer's theory of regularity structures, a revolutionary framework that has transformed the field. This final chapter, which constitutes an original research monograph, develops the abstract theory from first principles, proves the central Reconstruction Theorem, and demonstrates its application to the KPZ and  $\Phi_3^4$  equations. It provides a complete, thesis-level treatment of the subject, including a thorough discussion of the necessary renormalisation procedures.

By progressing from the classical to the singular, this thesis aims to provide a coherent and comprehensive account of the state of the art in the field of SPDEs. It showcases the development of a powerful new mathematical language and demonstrates its utility in solving long-standing open problems, thereby bridging the gap between physical intuition and mathematical rigour. The work presented here not only consolidates and builds upon existing knowledge but also provides a definitive and accessible entry point into one of the most exciting and rapidly developing areas of modern mathematics.



# Chapter 1

## Foundations of the Modern Theory of Partial Differential Equations

### 1 Introduction: A Historical and Conceptual Overview

The theory of Partial Differential Equations (PDEs) constitutes a principal pillar of modern mathematics, providing the language for a vast range of phenomena across the natural sciences, engineering, and even finance. Its development is a rich narrative of the symbiotic relationship between the quest to model the physical world and the pursuit of abstract mathematical truth. This chapter is dedicated to laying the foundational groundwork for the modern, analysis-based approach to PDEs, a framework that has proven indispensable for understanding the existence, uniqueness, and qualitative properties of solutions.

#### 1.1 The Genesis: From Vibrating Strings to Heat Flow

The origins of PDE theory can be traced to the mid-18th century, with the investigation of the vibrating string problem. The question was to describe the motion of a taut, flexible string fixed at both ends. In 1747, Jean le Rond d'Alembert derived the one-dimensional **wave equation**,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

and provided his celebrated solution,  $u(x, t) = f(x + ct) + g(x - ct)$ , representing the superposition of two waves travelling in opposite directions. This work sparked a vigorous debate with Leonhard Euler and Daniel Bernoulli regarding the nature of the initial functions  $f$  and  $g$ . Could they be any arbitrary function drawn by hand, or must they be analytic? This controversy was a harbinger of the central role that the concept of a ‘function’ would play in the development of analysis.

Decades later, in his seminal 1822 work *Théorie analytique de la chaleur*, Joseph Fourier introduced the **heat equation** to model the diffusion of thermal energy in a solid body:

$$\frac{\partial u}{\partial t} = k \Delta u \quad (2)$$

To solve this equation, Fourier developed the revolutionary technique of representing a function as an infinite series of trigonometric functions—what we now call a Fourier series. His audacious claim that *any* arbitrary function could be so represented was met with scepticism but ultimately laid the groundwork for harmonic analysis and the rigorous study of function spaces.

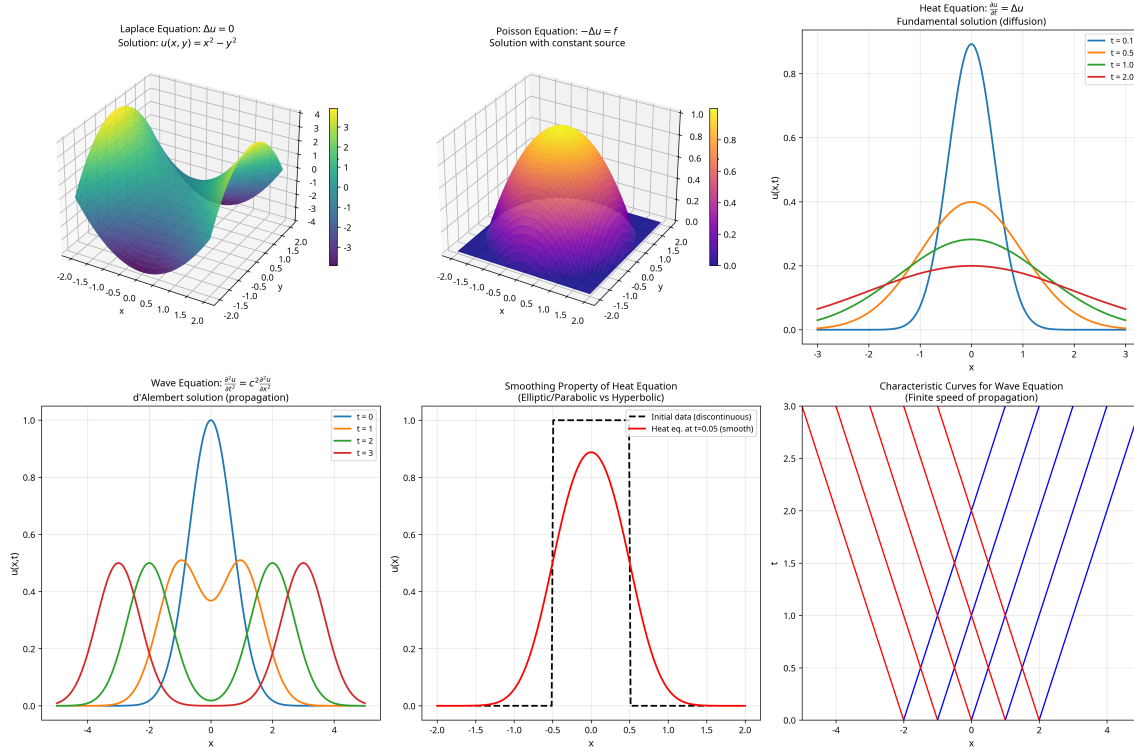
#### 1.2 The 19th Century: A Flourishing of Models

The 19th century witnessed the formulation of the cornerstone equations of mathematical physics. Pierre-Simon Laplace and Siméon Denis Poisson developed the **Laplace and Poisson equations**,

$$\Delta u = 0 \quad \text{and} \quad -\Delta u = f \quad (3)$$



to describe gravitational and electrostatic potentials. The study of these elliptic equations led to the development of potential theory and the discovery of the remarkable smoothness of harmonic functions. In fluid dynamics, Claude-Louis Navier and George Gabriel Stokes formulated the **Navier–Stokes equations**, a system of nonlinear PDEs describing the motion of viscous fluids. In electromagnetism, James Clerk Maxwell unified electricity, magnetism, and light with his celebrated system of equations, which are first-order linear PDEs.



**Figure 1: Classical Partial Differential Equations.** This comprehensive figure illustrates the three fundamental classes of PDEs that motivated the development of modern PDE theory. **(Top left)** Solution to Laplace’s equation  $\Delta u = 0$ , showing the harmonic function  $u(x, y) = x^2 - y^2$ , which exhibits the characteristic saddle-point geometry. **(Top middle)** Solution to the Poisson equation  $-\Delta u = f$  with a constant source term on a disk, demonstrating the smoothing effect of elliptic operators. **(Top right)** Fundamental solution of the heat equation  $\frac{\partial u}{\partial t} = \Delta u$ , showing the diffusion and spreading of an initial Gaussian profile over time. **(Bottom left)** d’Alembert’s solution to the wave equation  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ , illustrating the propagation of a wave pulse in both directions without dispersion. **(Bottom middle)** Comparison of the smoothing property: an initially discontinuous function is immediately smoothed by the heat equation, demonstrating the fundamental difference between parabolic and hyperbolic equations. **(Bottom right)** Characteristic curves for the wave equation, showing the finite speed of propagation along lines  $x \pm ct = \text{const}$ , a hallmark of hyperbolic PDEs.

### 1.3 The 20th Century Paradigm Shift: The Need for a Broader View

By the turn of the 20th century, it became increasingly clear that the classical framework of continuously differentiable solutions was inadequate. Physical phenomena such as shock waves in gas dynamics or the interfaces between different media required a more general notion of a solution. The mathematical world needed a way to make sense of equations whose solutions were not smooth.

This impetus led to one of the most profound developments in modern mathematics: the birth of **functional analysis**. The revolutionary idea was to treat functions themselves as points

in an infinite-dimensional vector space. This allowed the powerful geometric and algebraic tools of linear algebra to be generalised and applied to problems in analysis. The introduction of normed spaces, complete spaces (Banach spaces), and spaces with an inner product (Hilbert spaces) by mathematicians like Stefan Banach, David Hilbert, and Frigyes Riesz provided the abstract setting.

Within this framework, the concept of a **weak solution** emerged. The idea is to reformulate the PDE in an integral form, transferring derivatives from the unknown solution onto a space of infinitely smooth ‘test functions’. This was formalised by Sergei Sobolev in the 1930s with the introduction of what are now called **Sobolev spaces**. These spaces consist of functions that may not be differentiable in the classical sense but possess ‘weak derivatives’ that are well-behaved in an integral sense. Almost simultaneously, Laurent Schwartz developed the **theory of distributions**, or generalised functions, providing an even broader framework in which every distribution is infinitely differentiable.

This modern approach, which combines functional analysis and the theory of weak solutions, has been spectacularly successful. It provides a unified framework for proving the existence and uniqueness of solutions to a vast class of linear and nonlinear PDEs. This chapter is devoted to a systematic and rigorous development of these foundational concepts. We will build the theory from the ground up, starting with the essential structures of functional analysis and culminating in the Lax–Milgram theorem, a cornerstone result for the existence of weak solutions to elliptic equations. Throughout, we will use the classical equations of mathematical physics as motivating examples, demonstrating how the abstract theory provides deep and essential insights into the concrete problems from which it grew.

## 2 Functional Analytic Preliminaries: The Language of Modern PDE Theory

The shift from classical to modern PDE theory is fundamentally a change in language and perspective. The new language is that of functional analysis, and the new perspective is to view solutions not as individual functions but as points in an infinite-dimensional space. This section provides a rigorous, self-contained development of the essential concepts: normed vector spaces, Banach spaces, and the geometrically rich Hilbert spaces.

### 2.1 Normed Vector Spaces and Banach Spaces

We begin with the foundational algebraic and topological structure.

**Definition 2.1** (Normed Vector Space). A **normed vector space** is a pair  $(V, \|\cdot\|)$ , where  $V$  is a vector space over a field  $\mathbb{F}$  (which for our purposes will always be  $\mathbb{R}$  or  $\mathbb{C}$ ), and  $\|\cdot\| : V \rightarrow [0, \infty)$  is a function, called the **norm**, satisfying the following axioms for all  $u, v \in V$  and all scalars  $\alpha \in \mathbb{F}$ :

- (i) **Positive Definiteness:**  $\|u\| = 0$  if and only if  $u = 0$ .
- (ii) **Absolute Homogeneity:**  $\|\alpha u\| = |\alpha| \|u\|$ .
- (iii) **Triangle Inequality (Subadditivity):**  $\|u + v\| \leq \|u\| + \|v\|$ .

The norm introduces a natural notion of distance via the metric  $d(u, v) = \|u - v\|$ . This allows us to talk about convergence and continuity. A sequence  $(u_n)_{n=1}^\infty$  in  $V$  converges to  $u \in V$  if  $\lim_{n \rightarrow \infty} \|u_n - u\| = 0$ .

However, a normed space is not sufficient for analysis, which relies on limit processes. We need to ensure that sequences that *should* converge actually do converge to a point within the space. This is the property of completeness.



**Definition 2.2** (Cauchy Sequence and Banach Space). A sequence  $(u_n)_{n=1}^\infty$  in a normed space  $V$  is a **Cauchy sequence** if for every  $\epsilon > 0$ , there exists an integer  $N$  such that for all  $m, n > N$ , we have  $\|u_m - u_n\| < \epsilon$ . A normed vector space  $V$  is **complete** if every Cauchy sequence in  $V$  converges to a limit in  $V$ . A complete normed vector space is called a **Banach space**.

**Example 2.3** (The Space of Continuous Functions,  $C(\overline{\Omega})$ ). Let  $\Omega \subset \mathbb{R}^n$  be a bounded, open set. The space  $C(\overline{\Omega})$  consists of all continuous functions  $u : \overline{\Omega} \rightarrow \mathbb{R}$ . This is a vector space under the usual pointwise addition and scalar multiplication. We equip it with the **supremum norm** (or uniform norm):

$$\|u\|_{C(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} |u(x)| \quad (4)$$

This is a Banach space. The proof of completeness is a cornerstone of real analysis. Let  $(u_n)$  be a Cauchy sequence in  $C(\overline{\Omega})$ . For any fixed  $x \in \overline{\Omega}$ , the sequence of real numbers  $(u_n(x))$  is a Cauchy sequence in  $\mathbb{R}$ , since  $|u_m(x) - u_n(x)| \leq \|u_m - u_n\|_{C(\overline{\Omega})}$ . Since  $\mathbb{R}$  is complete, this sequence converges to a limit, which we define as  $u(x)$ . This defines a pointwise limit function  $u$ . We must show that this convergence is uniform and that the limit function  $u$  is continuous. Given  $\epsilon > 0$ , we can find  $N$  such that for all  $m, n > N$ ,  $\|u_m - u_n\|_{C(\overline{\Omega})} < \epsilon/2$ . Taking the limit as  $m \rightarrow \infty$ , we get  $|u(x) - u_n(x)| \leq \epsilon/2$  for all  $x$  and all  $n > N$ . This is the definition of uniform convergence. The uniform limit of a sequence of continuous functions is continuous, so  $u \in C(\overline{\Omega})$ . Thus,  $C(\overline{\Omega})$  is a Banach space.

**Example 2.4** (The Lebesgue Spaces,  $L^p(\Omega)$ ). While  $C(\overline{\Omega})$  is a natural space, it is often too restrictive for PDE theory. A much more flexible and powerful class of spaces are the Lebesgue spaces, which are built upon Lebesgue's theory of integration.

Let  $\Omega \subset \mathbb{R}^n$  be a measurable set. For  $1 \leq p < \infty$ , the **Lebesgue space**  $L^p(\Omega)$  is the space of all equivalence classes of measurable functions  $u : \Omega \rightarrow \mathbb{R}$  for which the  $p$ -norm is finite:

$$\|u\|_{L^p(\Omega)} := \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p} < \infty \quad (5)$$

Two functions are considered equivalent if they are equal *almost everywhere* (a.e.), i.e., they differ only on a set of measure zero. This is necessary to ensure the positive definiteness of the norm.

For  $p = \infty$ , the space  $L^\infty(\Omega)$  consists of essentially bounded functions, with the norm:

$$\|u\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |u(x)| = \inf\{M \geq 0 \mid |u(x)| \leq M \text{ for a.e. } x \in \Omega\} \quad (6)$$

The fundamental result concerning these spaces is their completeness.

**Theorem 2.5** (Riesz–Fischer). *For any measurable set  $\Omega$  and any  $1 \leq p < \infty$ , the space  $L^p(\Omega)$  is a Banach space.*

*Proof.* Let  $(u_n)_{n=1}^\infty$  be a Cauchy sequence in  $L^p(\Omega)$ . The core of the proof is to show that this sequence has a subsequence that converges pointwise almost everywhere to a function  $u$ , and that this function  $u$  is in  $L^p(\Omega)$  and is the limit of the original sequence in the  $L^p$  norm.

**Step 1: Constructing a rapidly converging subsequence.** Since  $(u_n)$  is Cauchy, we can choose a subsequence  $(u_{n_k})$  such that  $\|u_{n_{k+1}} - u_{n_k}\|_{L^p} < 1/2^k$  for all  $k \geq 1$ . Let  $g_k = u_{n_k}$  and define the partial sums  $S_K(x) = \sum_{k=1}^K |g_{k+1}(x) - g_k(x)|$ . By the triangle inequality (Minkowski's inequality) in  $L^p$ ,

$$\|S_K\|_{L^p} \leq \sum_{k=1}^K \|g_{k+1} - g_k\|_{L^p} < \sum_{k=1}^K \frac{1}{2^k} < 1 \quad (7)$$

Let  $S(x) = \lim_{K \rightarrow \infty} S_K(x) = \sum_{k=1}^\infty |g_{k+1}(x) - g_k(x)|$ . By the Monotone Convergence Theorem,  $\|S\|_{L^p}^p = \int \lim_{K \rightarrow \infty} S_K^p dx = \lim_{K \rightarrow \infty} \int S_K^p dx = \lim_{K \rightarrow \infty} \|S_K\|_{L^p}^p \leq 1$ . Thus,  $S \in L^p(\Omega)$ , which implies that  $S(x)$  is finite for almost every  $x \in \Omega$ .

**Step 2: Pointwise convergence of the subsequence.** The series  $\sum_{k=1}^{\infty} (g_{k+1}(x) - g_k(x))$  converges absolutely for almost every  $x$ , because the sum of the absolute values is  $S(x)$ , which is finite a.e. This is a telescoping series, and its convergence implies that the sequence  $g_k(x)$  converges for almost every  $x$ . Let us define  $u(x) = g_1(x) + \sum_{k=1}^{\infty} (g_{k+1}(x) - g_k(x))$  for those  $x$  where the series converges, and  $u(x) = 0$  otherwise. Then  $g_k(x) \rightarrow u(x)$  a.e.

**Step 3: Showing  $u \in L^p(\Omega)$  and convergence in norm.** We have  $|g_k(x)| \leq |g_1(x)| + \sum_{j=1}^{k-1} |g_{j+1}(x) - g_j(x)| \leq |g_1(x)| + S(x)$ . Since  $g_1$  and  $S$  are in  $L^p$ , so is their sum. Taking the limit as  $k \rightarrow \infty$ , we have  $|u(x)| \leq |g_1(x)| + S(x)$  a.e. This implies that  $u \in L^p(\Omega)$ .

Now we show that the subsequence converges in norm to  $u$ . We have  $|g_k(x) - u(x)| \rightarrow 0$  a.e. Also,  $|g_k(x) - u(x)|^p \leq (2(|g_1(x)| + S(x)))^p$ , which is an integrable function. By the Dominated Convergence Theorem,

$$\lim_{k \rightarrow \infty} \|g_k - u\|_{L^p}^p = \lim_{k \rightarrow \infty} \int_{\Omega} |g_k(x) - u(x)|^p dx = \int_{\Omega} \lim_{k \rightarrow \infty} |g_k(x) - u(x)|^p dx = 0 \quad (8)$$

So, the subsequence  $(g_k)$  converges to  $u$  in  $L^p$ .

**Step 4: Convergence of the original sequence.** Finally, we use the triangle inequality to show the original Cauchy sequence  $(u_n)$  converges to  $u$ . Given  $\epsilon > 0$ , since  $(u_n)$  is Cauchy, there is an  $N$  such that  $\|u_n - u_m\|_{L^p} < \epsilon/2$  for  $n, m > N$ . Since the subsequence  $(u_{n_k})$  converges to  $u$ , we can choose  $k$  large enough so that  $n_k > N$  and  $\|u_{n_k} - u\|_{L^p} < \epsilon/2$ . Then for any  $n > N$ ,

$$\|u_n - u\|_{L^p} \leq \|u_n - u_{n_k}\|_{L^p} + \|u_{n_k} - u\|_{L^p} < \epsilon/2 + \epsilon/2 = \epsilon \quad (9)$$

Thus, the original sequence converges to  $u$  in  $L^p(\Omega)$ .  $\square$

Two inequalities are fundamental to the study of  $L^p$  spaces.

**Theorem 2.6** (Hölder's Inequality). *Let  $1 \leq p \leq \infty$  and let  $q$  be the **conjugate exponent** to  $p$ , defined by  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $u \in L^p(\Omega)$  and  $v \in L^q(\Omega)$ , then their product  $uv$  is in  $L^1(\Omega)$ , and*

$$\int_{\Omega} |u(x)v(x)| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \quad (10)$$

*Proof.* The cases  $p = 1, q = \infty$  (or vice-versa) are straightforward. For  $1 < p < \infty$ , the proof relies on Young's inequality: for non-negative  $a, b$ , we have  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ . Let  $\tilde{u} = u/\|u\|_{L^p}$  and  $\tilde{v} = v/\|v\|_{L^q}$ . Applying Young's inequality with  $a = |\tilde{u}(x)|$  and  $b = |\tilde{v}(x)|$  and integrating over  $\Omega$  gives:

$$\int_{\Omega} |\tilde{u}(x)\tilde{v}(x)| dx \leq \int_{\Omega} \left( \frac{|\tilde{u}(x)|^p}{p} + \frac{|\tilde{v}(x)|^q}{q} \right) dx = \frac{1}{p} \|\tilde{u}\|_{L^p}^p + \frac{1}{q} \|\tilde{v}\|_{L^q}^q = \frac{1}{p} + \frac{1}{q} = 1 \quad (11)$$

Substituting back the definitions of  $\tilde{u}$  and  $\tilde{v}$  yields the result.  $\square$

**Theorem 2.7** (Minkowski's Inequality). *For  $1 \leq p \leq \infty$ , if  $u, v \in L^p(\Omega)$ , then their sum  $u + v$  is also in  $L^p(\Omega)$ , and*

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \quad (12)$$

This is the triangle inequality for the  $L^p$  norm. The proof for  $1 < p < \infty$  uses Hölder's inequality:

$$\|u + v\|_{L^p}^p = \int |u + v|^p dx \leq \int |u| |u + v|^{p-1} dx + \int |v| |u + v|^{p-1} dx \quad (13)$$

Applying Hölder's inequality to the first integral with exponent  $p$  for  $|u|$  and conjugate exponent  $q = p/(p-1)$  for  $|u + v|^{p-1}$ , we get:

$$\int |u| |u + v|^{p-1} dx \leq \|u\|_{L^p} \|u + v\|_{L^p}^{p-1} \quad (14)$$

Doing the same for the second integral and combining gives  $\|u + v\|_{L^p}^p \leq (\|u\|_{L^p} + \|v\|_{L^p}) \|u + v\|_{L^p}^{p-1}$ . Dividing by  $\|u + v\|_{L^p}^{p-1}$  yields the result.



## 2.2 Hilbert Spaces: The Geometry of $L^2$

Among the Banach spaces, one class stands out for its rich geometric structure: the Hilbert spaces. These are Banach spaces whose norm is induced by an inner product.

**Definition 2.8** (Inner Product and Hilbert Space). An **inner product** on a vector space  $V$  is a function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$  satisfying for all  $u, v, w \in V$  and  $\alpha \in \mathbb{F}$ :

- (i) **Conjugate Symmetry:**  $\langle u, v \rangle = \overline{\langle v, u \rangle}$ .
- (ii) **Linearity in the first argument:**  $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$ .
- (iii) **Positive Definiteness:**  $\langle u, u \rangle \geq 0$ , and  $\langle u, u \rangle = 0$  if and only if  $u = 0$ .

An inner product space is a **Hilbert space** if it is complete with respect to the norm induced by the inner product,  $\|u\| := \sqrt{\langle u, u \rangle}$ .

The space  $L^2(\Omega)$  is the canonical example of a Hilbert space in PDE theory, with the inner product:

$$\langle u, v \rangle_{L^2(\Omega)} := \int_{\Omega} u(x) \overline{v(x)} dx \quad (15)$$

(For real-valued functions, the complex conjugate is unnecessary). The Cauchy–Schwarz inequality is a direct consequence of the properties of the inner product and is a special case of Hölder’s inequality for  $p = q = 2$ :

**Theorem 2.9** (Cauchy–Schwarz Inequality). *In any inner product space,  $|\langle u, v \rangle| \leq \|u\| \|v\|$ .*

Not every norm comes from an inner product. The distinguishing feature is the **Parallelogram Law**:

**Theorem 2.10** (Parallelogram Law). *A norm on a vector space is induced by an inner product if and only if it satisfies the parallelogram law:*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad (16)$$

One can check that the  $L^p$  norms satisfy this identity only when  $p = 2$ . This is what makes  $L^2$  so special. The inner product provides a notion of **orthogonality**: two vectors  $u, v$  are orthogonal if  $\langle u, v \rangle = 0$ . This allows for geometric constructions like orthogonal projections and orthonormal bases, which are fundamental to many solution techniques.

**Theorem 2.11** (Riesz Representation Theorem). *Let  $H$  be a Hilbert space. For every continuous linear functional  $L : H \rightarrow \mathbb{F}$ , there exists a unique vector  $f_L \in H$  such that*

$$L(u) = \langle u, f_L \rangle \quad \text{for all } u \in H \quad (17)$$

*Furthermore, the mapping  $L \mapsto f_L$  is an isometric isomorphism between the dual space  $H^*$  and  $H$  itself, i.e.,  $\|L\|_{H^*} = \|f_L\|_H$ .*

*Proof Sketch.* The proof is constructive. If  $L = 0$ , we take  $f_L = 0$ . If  $L \neq 0$ , consider the kernel (or null space)  $K = \ker(L)$ . Since  $L$  is continuous,  $K$  is a closed subspace of  $H$ . Since  $L \neq 0$ ,  $K$  is not all of  $H$ . By the Projection Theorem for Hilbert spaces, we can find a non-zero vector  $z_0 \in K^\perp$ , the orthogonal complement of  $K$ . We can normalise  $z_0$  so that  $\|z_0\| = 1$ . We then seek  $f_L$  of the form  $f_L = \alpha z_0$ . We need to find  $\alpha$  such that  $L(u) = \langle u, \alpha z_0 \rangle = \overline{\alpha} \langle u, z_0 \rangle$ . This must hold for all  $u$ . For  $u = z_0$ , we get  $L(z_0) = \overline{\alpha} \|z_0\|^2 = \overline{\alpha}$ . So we must have  $\alpha = \overline{L(z_0)}$ . We then define  $f_L = \overline{L(z_0)} z_0$ . One can then verify that this choice works for all  $u \in H$ . Uniqueness follows from the fact that if  $\langle u, f_1 \rangle = \langle u, f_2 \rangle$  for all  $u$ , then  $\langle u, f_1 - f_2 \rangle = 0$ , and taking  $u = f_1 - f_2$  implies  $f_1 = f_2$ .  $\square$

This theorem is of paramount importance. It identifies a Hilbert space with its own dual space, and it is the key ingredient in the proof of the Lax–Milgram theorem, which is the main tool for establishing the existence of weak solutions to elliptic PDEs.

### 3 Generalising Differentiation: The Theory of Distributions

The classical definition of a derivative, which requires the evaluation of a limit of difference quotients, is fundamentally a pointwise concept. As we have seen, this is too restrictive for many problems in PDE theory where solutions are not expected to be smooth. The theory of distributions, or generalised functions, developed by Laurent Schwartz in the 1940s, provides a powerful and elegant way to differentiate a much broader class of objects.

#### 3.1 The Space of Test Functions

The central idea of distribution theory is to redefine functions and their derivatives not by their pointwise values, but by how they act on a space of infinitely smooth, localised “test functions”.

**Definition 3.1** (Space of Test Functions). Let  $\Omega \subset \mathbb{R}^n$  be an open set. The space of **test functions** on  $\Omega$ , denoted  $\mathcal{D}(\Omega)$  or  $C_c^\infty(\Omega)$ , is the vector space of all infinitely differentiable functions  $\phi : \Omega \rightarrow \mathbb{R}$  that have compact support in  $\Omega$ . The **support** of a function,  $\text{supp}(\phi)$ , is the closure of the set of points where the function is non-zero.

The requirement of compact support means that each test function vanishes outside some bounded closed set contained within  $\Omega$ . This is crucial for ensuring that boundary terms in integration by parts always vanish.

#### 3.2 Distributions

With this space of test functions, we can now define distributions as continuous linear functionals.

**Definition 3.2** (Distribution). A **distribution** on  $\Omega$  is a continuous linear functional on the space of test functions  $\mathcal{D}(\Omega)$ . The space of all distributions on  $\Omega$  is the dual space of  $\mathcal{D}(\Omega)$ , denoted by  $\mathcal{D}'(\Omega)$ .

We use the bracket notation  $\langle T, \phi \rangle$  to denote the action of a distribution  $T \in \mathcal{D}'(\Omega)$  on a test function  $\phi \in \mathcal{D}(\Omega)$ . Linearity means  $\langle T, a\phi_1 + b\phi_2 \rangle = a\langle T, \phi_1 \rangle + b\langle T, \phi_2 \rangle$ . Continuity means that if  $\phi_k \rightarrow 0$  in  $\mathcal{D}(\Omega)$ , then  $\langle T, \phi_k \rangle \rightarrow 0$  in  $\mathbb{R}$ .

Any locally integrable function  $u \in L_{\text{loc}}^1(\Omega)$  can be identified with a **regular distribution**  $T_u$  via integration:

$$\langle T_u, \phi \rangle := \int_{\Omega} u(x)\phi(x)dx, \quad \forall \phi \in \mathcal{D}(\Omega) \quad (18)$$

**Example 3.3** (The Dirac Delta Distribution). The most famous distribution that is not regular is the **Dirac delta distribution**,  $\delta_{x_0}$ , centred at a point  $x_0 \in \Omega$ . It is defined by its action of evaluating a test function at  $x_0$ :

$$\langle \delta_{x_0}, \phi \rangle := \phi(x_0) \quad (19)$$

This is clearly a linear functional. It is also continuous. However, one can prove that there is no function  $u \in L_{\text{loc}}^1(\Omega)$  such that  $\int u(x)\phi(x)dx = \phi(x_0)$  for all test functions. Thus,  $\delta_{x_0}$  is a **singular distribution**.

#### 3.3 Differentiation of Distributions

The true power of this framework is revealed in how it defines differentiation. We can define the derivative of *any* distribution, regardless of its regularity.

**Definition 3.4** (Derivative of a Distribution). Let  $T \in \mathcal{D}'(\Omega)$  and let  $\alpha$  be a multi-index. The  $\alpha$ -th partial derivative of  $T$ , denoted  $D^\alpha T$ , is the distribution defined by its action on test functions:

$$\langle D^\alpha T, \phi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega) \quad (20)$$



This definition is motivated by integration by parts. If  $u$  were a smooth function, we would have  $\int (D^\alpha u)\phi = (-1)^{|\alpha|} \int u(D^\alpha \phi)$ , since the boundary terms vanish due to the compact support of  $\phi$ . The definition generalises this formula to any distribution. A remarkable consequence is that every distribution is infinitely differentiable in the distributional sense.

**Example 3.5** (Derivative of the Heaviside Function). Let  $H$  be the Heaviside function in one dimension,  $H(x) = 1$  for  $x > 0$  and  $H(x) = 0$  for  $x < 0$ . It is not differentiable at  $x = 0$  in the classical sense. Its distributional derivative is the distribution  $H'$  whose action on a test function  $\phi \in \mathcal{D}(\mathbb{R})$  is:

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_0^{\infty} \phi'(x)dx = \phi(0) \quad (21)$$

This is precisely the definition of the Dirac delta distribution centred at 0. So, we can write the remarkable formula  $H' = \delta_0$ . The derivative of a jump discontinuity is a delta function, a concept that is meaningless in classical calculus but perfectly rigorous in the theory of distributions.

### 3.4 Weak Derivatives

The concept of a distributional derivative leads naturally to the definition of a weak derivative for functions.

**Definition 3.6** (Weak Derivative). Let  $u \in L^1_{\text{loc}}(\Omega)$ . A function  $v \in L^1_{\text{loc}}(\Omega)$  is the  $\alpha$ -th **weak partial derivative** of  $u$  if it satisfies:

$$\int_{\Omega} v(x)\phi(x)dx = (-1)^{|\alpha|} \int_{\Omega} u(x)(D^\alpha \phi)(x)dx, \quad \forall \phi \in \mathcal{D}(\Omega) \quad (22)$$

In other words, the weak derivative of  $u$  is a function  $v$  that represents the distributional derivative of  $u$ . If such a function  $v$  exists, we denote it by  $D^\alpha u$ .

**Example 3.7** (A Function with a Weak Derivative). Consider the function  $u(x) = |x|$  on  $\Omega = (-1, 1)$ . It is not differentiable at  $x = 0$ . Let's find its weak derivative. We are looking for a function  $v$  such that  $\int_{-1}^1 v\phi = -\int_{-1}^1 |x|\phi' dx$ . We split the integral:

$$-\int_{-1}^1 |x|\phi' dx = -\int_{-1}^0 (-x)\phi' dx - \int_0^1 x\phi' dx \quad (23)$$

Integrating by parts on each interval gives:

$$\int_{-1}^0 \phi(x)dx - \int_0^1 \phi(x)dx = \int_{-1}^1 \text{sgn}(x)\phi(x)dx \quad (24)$$

where  $\text{sgn}(x)$  is the sign function. By comparing this with the definition of the weak derivative, we see that the weak derivative of  $|x|$  is the function  $v(x) = \text{sgn}(x)$ , which is a perfectly well-defined function in  $L^\infty(-1, 1)$ . This demonstrates that a function can fail to be classically differentiable at a point but still possess a weak derivative that is a regular function.

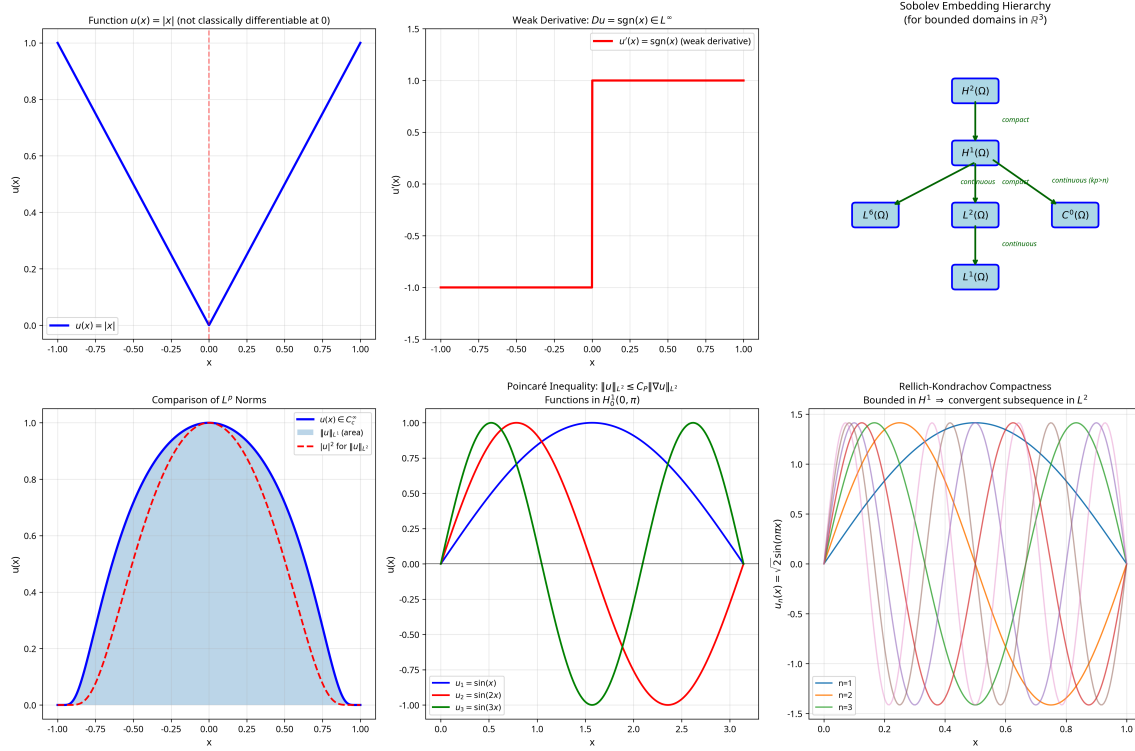


Figure 2: **Sobolev Spaces and Weak Derivatives.** This figure illustrates the key concepts of Sobolev space theory. **(Top left)** The function  $u(x) = |x|$ , which is not classically differentiable at  $x = 0$  but possesses a weak derivative. **(Top middle)** The weak derivative  $Du = \text{sgn}(x)$ , which is a well-defined function in  $L^\infty(-1, 1)$ , demonstrating that weak derivatives can exist even when classical derivatives do not. **(Top right)** Sobolev embedding hierarchy for bounded domains in  $\mathbb{R}^3$ , showing the relationships between various function spaces. Higher regularity spaces (e.g.,  $H^2$ ) embed into lower regularity spaces (e.g.,  $H^1$ ,  $L^2$ ), and when  $kp > n$ , Sobolev functions are continuous. **(Bottom left)** Comparison of  $L^p$  norms for a smooth, compactly supported function, illustrating how different norms measure different aspects of a function's size. **(Bottom middle)** Functions in  $H_0^1(0, \pi)$  that vanish at the boundaries, illustrating the Poincaré inequality, which states that the  $L^2$  norm can be controlled by the  $L^2$  norm of the gradient. **(Bottom right)** Illustration of the Rellich–Kondrachov compactness theorem: a bounded sequence in  $H^1$  (here, the eigenfunctions  $\sin(n\pi x)$ ) has a convergent subsequence in  $L^2$ , a fundamental tool for existence proofs.

## 4 Sobolev Spaces: The Modern Setting for PDEs

We have established the abstract framework of Banach and Hilbert spaces and generalised the notion of differentiation using distribution theory. We are now in a position to define the most important class of function spaces for the modern study of partial differential equations: the Sobolev spaces.

### 4.1 Definition and Properties

**Definition 4.1** (Sobolev Space  $W^{k,p}(\Omega)$ ). Let  $\Omega$  be an open set in  $\mathbb{R}^n$ , let  $k$  be a non-negative integer, and let  $1 \leq p \leq \infty$ . The **Sobolev space**  $W^{k,p}(\Omega)$  is defined as the set of all functions  $u \in L^p(\Omega)$  such that for every multi-index  $\alpha$  with  $|\alpha| \leq k$ , the weak derivative  $D^\alpha u$  exists and belongs to  $L^p(\Omega)$ .

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\} \quad (25)$$

This space is equipped with the norm:

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty \quad (26)$$

A fundamental theorem, which we state without proof, is that these spaces are complete.

**Theorem 4.2** (Completeness of Sobolev Spaces). *For any  $k \geq 0$  and  $1 \leq p \leq \infty$ , the Sobolev space  $W^{k,p}(\Omega)$  is a Banach space.*

In the special case where  $p = 2$ , the Sobolev spaces are Hilbert spaces, denoted by  $H^k(\Omega) = W^{k,2}(\Omega)$ . The inner product is given by:

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} (D^\alpha u)(x) (D^\alpha v)(x) dx \quad (27)$$

## 4.2 Boundary Values and the Spaces $H_0^k(\Omega)$

When solving PDEs, we almost always have to impose boundary conditions. In the variational framework, this is handled by working in a subspace of the full Sobolev space.

**Definition 4.3** (The Space  $W_0^{k,p}(\Omega)$ ). The space  $W_0^{k,p}(\Omega)$  is defined as the closure of the space of test functions  $C_c^\infty(\Omega)$  with respect to the  $W^{k,p}(\Omega)$  norm. The corresponding Hilbert space for  $p = 2$  is denoted  $H_0^k(\Omega)$ .

## 4.3 The Sobolev Embedding Theorems

One of the most powerful aspects of Sobolev space theory is the collection of results known as the **Sobolev embedding theorems**. These theorems provide precise information about the regularity of functions in a Sobolev space.

**Theorem 4.4** (Sobolev Embedding Theorem). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a sufficiently smooth boundary.*

1. *If  $kp < n$ , then  $W^{k,p}(\Omega)$  is continuously embedded in  $L^q(\Omega)$  for all  $1 \leq q \leq p^*$ , where  $p^* = \frac{np}{n-kp}$  is the **Sobolev conjugate exponent**.*
2. *If  $kp = n$ , then  $W^{k,p}(\Omega)$  is continuously embedded in  $L^q(\Omega)$  for all  $1 \leq q < \infty$ .*
3. *If  $kp > n$ , then  $W^{k,p}(\Omega)$  is continuously embedded in the space of Hölder continuous functions  $C^{0,\gamma}(\overline{\Omega})$  for some  $\gamma > 0$ . In particular, the functions are continuous and bounded.*

**Theorem 4.5** (Rellich–Kondrachov Compactness Theorem). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary. If  $kp < n$ , the embedding of  $W^{k,p}(\Omega)$  into  $L^q(\Omega)$  is **compact** for all  $1 \leq q < p^*$ . If  $kp > n$ , the embedding into  $C(\overline{\Omega})$  is also compact.*

## 4.4 The Poincaré Inequality

A crucial tool for establishing the coercivity of bilinear forms associated with elliptic operators is the Poincaré inequality.

**Theorem 4.6** (Poincaré Inequality). *Let  $\Omega$  be a bounded, connected open set in  $\mathbb{R}^n$ . Then there exists a constant  $C_P > 0$ , depending only on  $\Omega$ , such that for every function  $u \in H_0^1(\Omega)$ ,*

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)} \quad (28)$$



*Proof Sketch.* We prove this for the case where  $\Omega$  is bounded in the  $x_1$  direction, say  $\Omega \subset \{x \in \mathbb{R}^n | a < x_1 < b\}$ . For any  $u \in C_c^\infty(\Omega)$ , we can write for any  $x = (x_1, \dots, x_n) \in \Omega$ :

$$u(x) = \int_a^{x_1} \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) dy_1 \quad (29)$$

By the Cauchy–Schwarz inequality for integrals,

$$|u(x)|^2 \leq (b-a) \int_a^{x_1} |\nabla u|^2 dy_1 \quad (30)$$

Integrating over  $\Omega$  gives  $\|u\|_{L^2} \leq (b-a)\|\nabla u\|_{L^2}$ . The result for general  $u \in H_0^1(\Omega)$  follows by density.  $\square$

## 5 The Lax–Milgram Theorem and Elliptic Equations

We now have all the tools to prove the main existence theorem for linear elliptic PDEs.

**Theorem 5.1** (Lax–Milgram). *Let  $H$  be a Hilbert space,  $a : H \times H \rightarrow \mathbb{R}$  a bilinear form, and  $L : H \rightarrow \mathbb{R}$  a linear functional. Suppose that:*

- (i)  *$a$  is **continuous**: there exists  $M > 0$  such that  $|a(u, v)| \leq M\|u\|_H\|v\|_H$  for all  $u, v \in H$ .*
- (ii)  *$a$  is **coercive**: there exists  $\alpha > 0$  such that  $a(u, u) \geq \alpha\|u\|_H^2$  for all  $u \in H$ .*
- (iii)  *$L$  is **continuous**: there exists  $C > 0$  such that  $|L(v)| \leq C\|v\|_H$  for all  $v \in H$ .*

*Then there exists a unique  $u \in H$  such that*

$$a(u, v) = L(v) \quad \forall v \in H \quad (31)$$

*Proof Sketch.* The proof uses the Riesz Representation Theorem to associate the bilinear form  $a(u, v)$  with an operator  $A : H \rightarrow H$  such that  $a(u, v) = \langle Au, v \rangle$ . The problem becomes solving  $Au = f$  for some  $f \in H$ . The coercivity of  $a$  is used to show that  $A$  is invertible, guaranteeing a unique solution.  $\square$

**Example 5.2** (The Dirichlet Problem for the Poisson Equation). Consider  $-\Delta u = f$  in  $\Omega$  with  $u = 0$  on  $\partial\Omega$ . We seek a weak solution in  $H_0^1(\Omega)$ . The weak formulation is: find  $u \in H_0^1(\Omega)$  such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega) \quad (32)$$

We define  $a(u, v) = \int \nabla u \cdot \nabla v$  and  $L(v) = \int f v$ . Continuity of  $a$  and  $L$  follows from Cauchy–Schwarz. Coercivity of  $a$  follows from the Poincaré inequality:  $a(u, u) = \|\nabla u\|_{L^2}^2 \geq \frac{1}{C_P^2} \|u\|_{L^2}^2$ . This gives  $a(u, u) \geq \alpha\|u\|_{H^1}^2$ . The Lax–Milgram theorem then guarantees a unique weak solution.

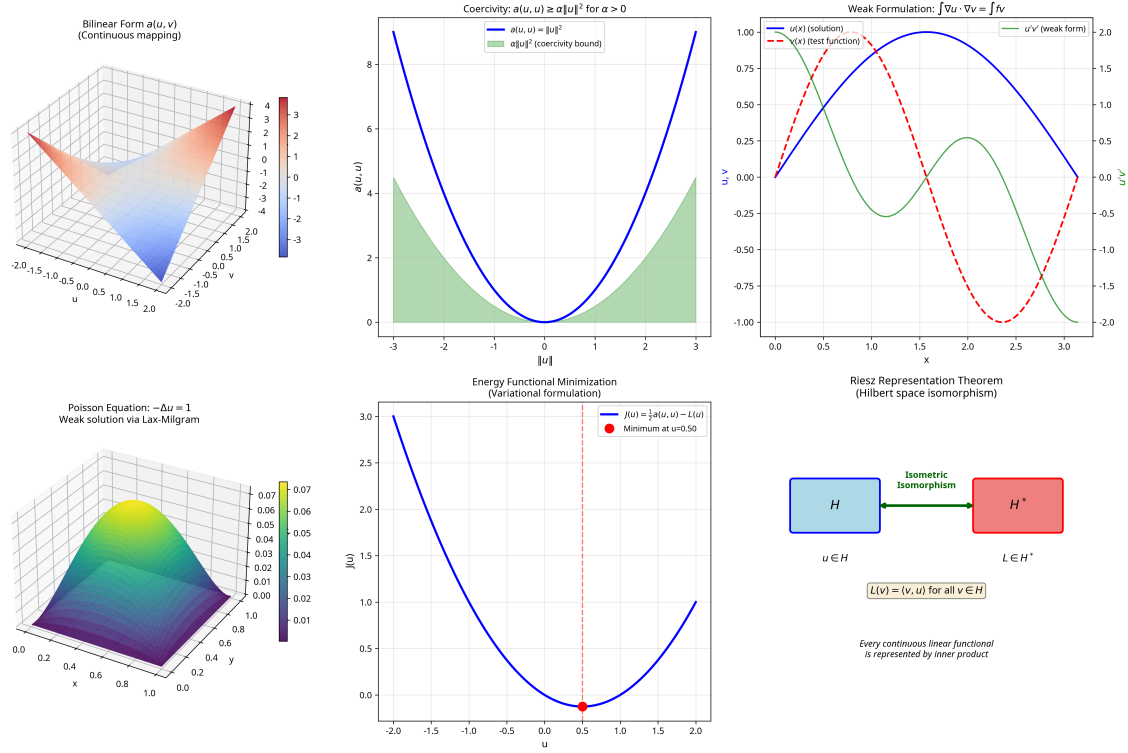


Figure 3: **Lax–Milgram Theorem and Variational Formulation.** This figure illustrates the abstract framework for solving elliptic PDEs via the Lax–Milgram theorem. **(Top left)** Visualisation of a bilinear form  $a(u, v)$  as a continuous mapping from  $H \times H \rightarrow \mathbb{R}$ . **(Top middle)** The coercivity condition  $a(u, u) \geq \alpha \|u\|^2$ , which ensures that the bilinear form is bounded below by a positive multiple of the square of the norm. This is the key to invertibility. **(Top right)** The weak formulation of a PDE, showing the relationship between the solution  $u$ , test function  $v$ , and their derivatives. The weak form  $\int \nabla u \cdot \nabla v = \int f v$  is obtained by integration by parts from the strong form  $-\Delta u = f$ . **(Bottom left)** Solution to the Poisson equation  $-\Delta u = 1$  on the unit square with homogeneous Dirichlet boundary conditions, obtained via the Lax–Milgram theorem. **(Bottom middle)** The energy functional  $J(u) = \frac{1}{2}a(u, u) - L(u)$ , whose minimiser is the weak solution to the PDE. The variational formulation transforms the PDE problem into an optimisation problem. **(Bottom right)** Diagram illustrating the Riesz Representation Theorem, which establishes an isometric isomorphism between a Hilbert space  $H$  and its dual  $H^*$ . Every continuous linear functional  $L$  is represented by an inner product with a unique element  $u \in H$ . This theorem is the foundation of the Lax–Milgram theorem.

## 6 Spectral Theory of Compact Self-Adjoint Operators

For many linear PDEs, the solution can be represented as a series expansion in terms of eigenfunctions of the differential operator. The abstract framework for this is the spectral theory of compact operators.

**Definition 6.1** (Compact Operator). A linear operator  $K : H_1 \rightarrow H_2$  between Hilbert spaces is **compact** if it maps bounded sets in  $H_1$  to precompact sets in  $H_2$ . (A set is precompact if its closure is compact).

**Theorem 6.2** (Spectral Theorem for Compact Self-Adjoint Operators). *Let  $H$  be a separable Hilbert space and let  $K : H \rightarrow H$  be a compact, self-adjoint operator. Then there exists an orthonormal sequence  $(e_k)_{k=1}^{\infty}$  of eigenvectors of  $K$  and a corresponding sequence of real eigenvalues*

$(\lambda_k)_{k=1}^\infty$  with  $\lambda_k \rightarrow 0$  such that

$$Kx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k \quad \forall x \in H \quad (33)$$

**Application: The Inverse Laplacian.** Consider the operator  $A = -\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$ . Its inverse,  $A^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega)$ , which maps a source function  $f$  to the solution  $u$  of the Poisson equation, can be shown to be a compact operator from  $L^2(\Omega)$  to itself. It is also self-adjoint. Therefore, the spectral theorem applies. The eigenfunctions of  $A^{-1}$  are the same as the eigenfunctions of  $A$ , and the eigenvalues are the reciprocals. This guarantees the existence of an orthonormal basis of  $L^2(\Omega)$  consisting of eigenfunctions of the Laplacian.

## 7 Conclusion

This chapter has built the modern functional analytic framework for PDEs. We have journeyed from the classical equations to the abstract spaces where their solutions live. We defined weak derivatives, Sobolev spaces, and proved the fundamental existence theorem for linear elliptic equations. This foundation will allow us to tackle more complex problems, including time-dependent equations, nonlinear equations, and questions of solution regularity in the chapters to come.

## References

- [1] d'Alembert, J. le R. (1747). *Recherches sur la courbe que forme une corde tendue mise en vibration*. Histoire de l'Académie Royale des Sciences et Belles-Lettres de Berlin, vol. 3, pp. 214–219.
- [2] Fourier, J. (1822). *Théorie analytique de la chaleur*. Paris: Firmin Didot, Père et Fils.
- [3] Sobolev, S. L. (1938). On a theorem of functional analysis. *Matematicheskii Sbornik*, 4(46), 3, 471–497.
- [4] Lebesgue, H. (1902). *Intégrale, longueur, aire*. Annali di Matematica Pura ed Applicata, 7(1), 231–359.
- [5] Schwartz, L. (1950). *Théorie des distributions*. Paris: Hermann.
- [6] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.
- [7] Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer.



# Chapter 2

## Classical Theory of Elliptic Equations

### 1 Introduction: The Nature of Elliptic Problems

Having established the functional analytic foundations in the previous chapter, we now turn our attention to the first of the three major classes of partial differential equations: the elliptic equations. Elliptic PDEs are arguably the most fundamental, as they typically describe steady-state phenomena, equilibrium configurations, and potentials. They are characterised by a strong smoothing property: their solutions are often infinitely differentiable in the interior of the domain, even when the boundary data or source terms are irregular. This is in stark contrast to hyperbolic equations, which propagate singularities, and parabolic equations, which smooth solutions only in the forward time direction.

The archetypal elliptic equation is Laplace's equation,  $\Delta u = 0$ . Its inhomogeneous counterpart, the Poisson equation,  $-\Delta u = f$ , is equally fundamental. A general second-order linear elliptic operator  $L$  in a domain  $\Omega \subset \mathbb{R}^n$  takes the form:

$$Lu = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \quad (2.1)$$

The operator  $L$  is defined as **elliptic** at a point  $x$  if the matrix of coefficients of the second-order terms,  $A(x) = (a_{ij}(x))$ , is positive definite. That is, there exists a constant  $\theta > 0$  such that for all  $\xi \in \mathbb{R}^n$ ,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad (2.2)$$

This condition ensures that the operator behaves, in a local sense, like the Laplacian. The theory of elliptic equations is a rich and beautiful subject that combines techniques from classical analysis, potential theory, and functional analysis. This chapter focuses on the *classical* theory, which seeks to establish the existence, uniqueness, and regularity of solutions that are continuously differentiable up to a certain order (i.e., classical solutions).

We will begin by exploring the fundamental properties of harmonic functions, which are the solutions to Laplace's equation. We will prove the Mean Value Property and the Maximum Principle, two cornerstone results that reveal the deep geometric and analytic structure of elliptic problems. We will then develop the theory of Green's functions, which provide an integral representation for the solution of the Poisson equation. This will lead us to the classical method of Perron for constructing solutions to the Dirichlet problem for Laplace's equation.

Finally, we will touch upon the regularity of solutions. A central theme in elliptic theory is that solutions are often much smoother than one might initially expect. We will discuss the concept of elliptic regularity, which states that if the coefficients of the operator and the source term are smooth, then any weak solution is also smooth. This theory culminates in the celebrated Schauder estimates, which provide a priori bounds on the norms of solutions in Hölder spaces and are a key tool in proving the existence of classical solutions to general linear elliptic equations.

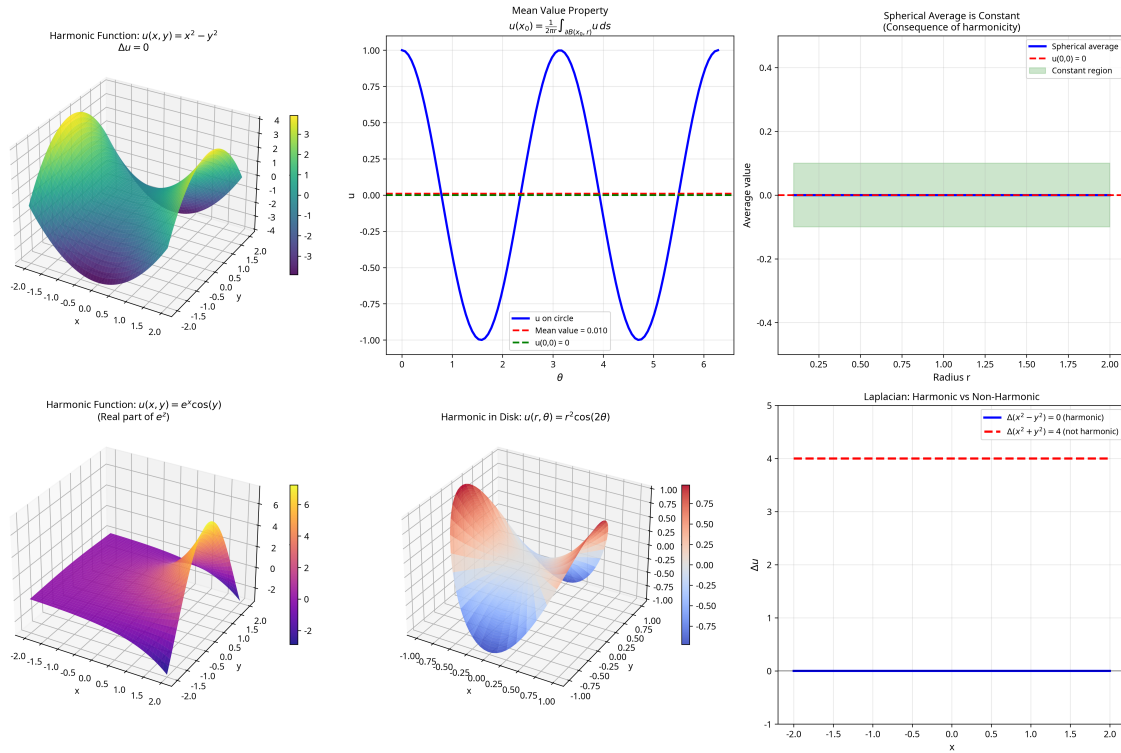
## 2 Fundamental Properties of Harmonic Functions

We begin our study of elliptic equations by examining the solutions to the most fundamental elliptic PDE: Laplace's equation,  $\Delta u = 0$ . Functions that satisfy Laplace's equation in a domain  $\Omega$  are called **harmonic functions**.

**Definition 2.1** (Harmonic Function). A function  $u \in C^2(\Omega)$  is said to be **harmonic** in an open set  $\Omega \subset \mathbb{R}^n$  if it satisfies Laplace's equation:

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0 \quad \text{for all } x \in \Omega \quad (2.3)$$

Harmonic functions possess a remarkable array of properties that are not shared by solutions to other types of PDEs. These properties reveal a deep connection between the Laplacian operator, geometry, and complex analysis. Two of the most important are the mean value property and the maximum principle.



**Figure 1: Harmonic Functions and the Mean Value Property.** This comprehensive figure illustrates the fundamental properties of harmonic functions. **(Top left)** A harmonic function  $u(x,y) = x^2 - y^2$ , which is the real part of the complex analytic function  $z^2$ . The function satisfies  $\Delta u = 0$  everywhere. **(Top middle)** Illustration of the mean value property: the value of a harmonic function at the centre of a circle equals the average of its values on the circle. For  $u(x,y) = x^2 - y^2$  at the origin, both the centre value and the mean are zero. **(Top right)** The spherical average of a harmonic function is constant as a function of the radius, a direct consequence of harmonicity. **(Bottom left)** Another example of a harmonic function:  $u(x,y) = e^x \cos(y)$ , the real part of  $e^z$ . **(Bottom middle)** A harmonic function in polar coordinates,  $u(r,\theta) = r^2 \cos(2\theta)$ , demonstrating the variety of forms harmonic functions can take. **(Bottom right)** Comparison of the Laplacian for harmonic and non-harmonic functions, showing that  $\Delta u = 0$  is the defining characteristic.

## 2.1 The Mean Value Property

The mean value property states that the value of a harmonic function at the centre of a ball is equal to the average of its values over the surface of that ball. This is a powerful rigidity property and is, in fact, equivalent to being harmonic.

**Theorem 2.2** (Mean Value Property for Harmonic Functions). *If  $u \in C^2(\Omega)$  is a harmonic function in a domain  $\Omega \subset \mathbb{R}^n$ , then for any ball  $B(x, r)$  such that  $\overline{B(x, r)} \subset \Omega$ , we have:*

$$u(x) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS(y) \quad (2.4)$$

where  $|\partial B(x, r)| = n\alpha(n)r^{n-1}$  is the surface area of the sphere  $\partial B(x, r)$ , and  $\alpha(n)$  is the volume of the unit ball in  $\mathbb{R}^n$ .

*Proof.* Let  $\phi(r)$  be the average value of  $u$  over the sphere  $\partial B(x, r)$ :

$$\phi(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x, r)} u(y) dS(y) \quad (2.5)$$

We want to show that  $\phi(r)$  is constant for  $0 < r < R$  (for some  $R$  such that  $\overline{B(x, R)} \subset \Omega$ ) and that its value is  $u(x)$ . We will do this by showing that  $\phi'(r) = 0$ .

We make a change of variables in the integral. Let  $y = x + rz$  where  $z \in \partial B(0, 1)$ . Then  $dS(y) = r^{n-1}dS(z)$ .

$$\phi(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} u(x + rz) dS(z) \quad (2.6)$$

Now, we differentiate with respect to  $r$  under the integral sign:

$$\phi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} \nabla u(x + rz) \cdot z dS(z) \quad (2.7)$$

We can rewrite the integrand using the change of variables  $y = x + rz$ . The outer unit normal to the sphere  $\partial B(x, r)$  at the point  $y$  is  $\nu(y) = \frac{y-x}{r} = z$ . So,  $\nabla u(y) \cdot z = \nabla u(y) \cdot \nu(y) = \frac{\partial u}{\partial \nu}(y)$ .

Returning to the original variables, we have:

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x, r)} \frac{\partial u}{\partial \nu}(y) dS(y) \quad (2.8)$$

Now we apply the Divergence Theorem to the vector field  $\nabla u$  over the ball  $B(x, r)$ :

$$\int_{\partial B(x, r)} \frac{\partial u}{\partial \nu}(y) dS(y) = \int_{B(x, r)} \operatorname{div}(\nabla u)(y) dy = \int_{B(x, r)} \Delta u(y) dy \quad (2.9)$$

Since  $u$  is harmonic,  $\Delta u = 0$  everywhere in  $\Omega$ , and thus in  $B(x, r)$ . Therefore, the integral is zero. This implies that  $\phi'(r) = 0$  for all  $r$  such that the ball is contained in  $\Omega$ . Thus,  $\phi(r)$  is a constant.

To find the value of this constant, we take the limit as  $r \rightarrow 0$ :

$$\lim_{r \rightarrow 0} \phi(r) = \lim_{r \rightarrow 0} \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS(y) \quad (2.10)$$

Since  $u$  is continuous, as  $r \rightarrow 0$ , the values of  $u(y)$  on the sphere  $\partial B(x, r)$  approach  $u(x)$ . The average value must therefore also approach  $u(x)$ . So,  $\lim_{r \rightarrow 0} \phi(r) = u(x)$ .

Since  $\phi(r)$  is constant, we must have  $\phi(r) = u(x)$  for all valid  $r$ . This completes the proof.  $\square$

It is a remarkable fact that the converse is also true: any continuous function that satisfies the mean value property in a domain must be harmonic in that domain. This shows that the mean value property is a complete characterisation of harmonicity.

## 2.2 The Maximum Principle

The maximum principle is one of the most important and powerful tools in the theory of elliptic equations. It states that a non-constant harmonic function cannot attain its maximum or minimum value in the interior of its domain. The maximum and minimum must occur on the boundary.

**Theorem 2.3** (Strong Maximum Principle). *Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded, and connected domain. Suppose  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is a harmonic function in  $\Omega$ .*

(i) *Then the maximum and minimum values of  $u$  are attained on the boundary  $\partial\Omega$ :*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad \text{and} \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u \quad (2.11)$$

(ii) *Furthermore, if  $u$  attains its maximum or minimum at an interior point of  $\Omega$ , then  $u$  must be a constant function throughout  $\Omega$ .*

*Proof.* We will prove the statement for the maximum; the proof for the minimum is analogous (or can be obtained by applying the maximum principle to  $-u$ ).

Let  $M = \max_{\overline{\Omega}} u$ . We want to show that if there exists a point  $x_0 \in \Omega$  such that  $u(x_0) = M$ , then  $u(x) = M$  for all  $x \in \Omega$ .

Let  $A = \{x \in \Omega \mid u(x) = M\}$ . By the continuity of  $u$ , the set  $A$  is a closed subset of  $\Omega$ . We will now show that  $A$  is also an open subset of  $\Omega$ .

Suppose  $x_0 \in A$ . Since  $\Omega$  is open, there exists a ball  $B(x_0, r)$  such that  $\overline{B(x_0, r)} \subset \Omega$ . By the Mean Value Property,

$$u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} u(y) dy \quad (2.12)$$

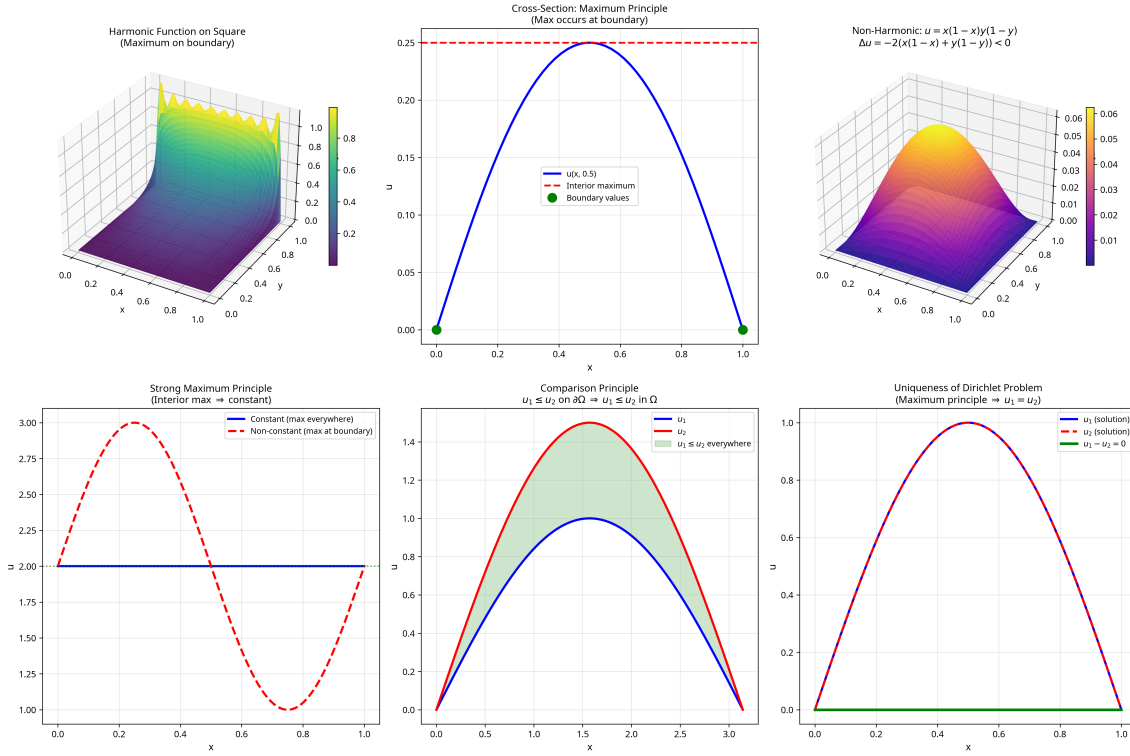
(Here we use the volume average, which is equivalent to the surface average). Since  $u(x_0) = M$  and  $u(y) \leq M$  for all  $y \in B(x_0, r)$ , this equation can only hold if  $u(y) = M$  for almost every  $y \in B(x_0, r)$ . Since  $u$  is continuous, this implies that  $u(y) = M$  for all  $y \in B(x_0, r)$ .

This means that the entire ball  $B(x_0, r)$  is contained in the set  $A$ . This proves that  $A$  is an open set.

So,  $A$  is a non-empty (since we assumed  $x_0 \in A$ ), open, and closed subset of the connected set  $\Omega$ . The only such subset is  $\Omega$  itself. Therefore,  $A = \Omega$ , which means that  $u(x) = M$  for all  $x \in \Omega$ .

This proves part (ii). Part (i) follows directly. If  $u$  is not constant, its maximum cannot be in the interior, so it must be on the boundary. If  $u$  is constant, its maximum is everywhere, including the boundary.  $\square$





**Figure 2: The Maximum Principle for Harmonic Functions.** This figure demonstrates the maximum principle and its consequences. **(Top left)** A harmonic function on a square domain, showing that the maximum value occurs on the boundary. **(Top middle)** A cross-section of the harmonic function, clearly showing that the interior values are bounded by the boundary values. **(Top right)** A non-harmonic function  $u(x, y) = x(1 - x)y(1 - y)$  that vanishes on the boundary but is positive in the interior, violating the maximum principle (because  $\Delta u < 0$ ). **(Bottom left)** Illustration of the strong maximum principle: if a harmonic function attains its maximum at an interior point, it must be constant. **(Bottom middle)** The comparison principle: if two harmonic functions satisfy  $u_1 \leq u_2$  on the boundary, then  $u_1 \leq u_2$  throughout the domain. **(Bottom right)** Uniqueness of the Dirichlet problem via the maximum principle: if two solutions exist, their difference is a harmonic function that vanishes on the boundary, hence must be zero everywhere.

The maximum principle has several immediate and important consequences.

**Corollary 2.4** (Uniqueness for the Dirichlet Problem). *Let  $\Omega$  be a bounded, connected domain. The Dirichlet problem for Laplace's equation,*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (2.13)$$

*has at most one solution  $u \in C^2(\Omega) \cap C(\overline{\Omega})$ .*

*Proof.* Suppose  $u_1$  and  $u_2$  are two solutions. Let  $w = u_1 - u_2$ . Then  $w$  satisfies:

$$\begin{cases} \Delta w = \Delta u_1 - \Delta u_2 = 0 - 0 = 0 & \text{in } \Omega \\ w = u_1 - u_2 = g - g = 0 & \text{on } \partial\Omega \end{cases} \quad (2.14)$$

By the Maximum Principle, the maximum and minimum of  $w$  must be on the boundary  $\partial\Omega$ . But on the boundary,  $w = 0$ . Therefore, for all  $x \in \Omega$ , we have  $0 \leq w(x) \leq 0$ , which implies  $w(x) = 0$  for all  $x \in \Omega$ . Thus,  $u_1 = u_2$ .  $\square$

### 3 Green's Functions and Representation Formulae

While the maximum principle guarantees the uniqueness of solutions to the Dirichlet problem, it does not provide a method for constructing them. The theory of Green's functions provides a powerful tool for obtaining explicit integral representation formulas for the solutions.

The idea is to find a *fundamental solution* for the Laplacian, which represents the potential generated by a point source. The solution to the Poisson equation can then be represented as a convolution of the source term with this fundamental solution. However, the fundamental solution does not, in general, satisfy the boundary conditions. The Green's function is a modification of the fundamental solution that is tailored to the specific geometry of the domain and satisfies the desired boundary conditions.

#### 3.1 The Fundamental Solution

We seek a solution to the equation  $-\Delta u = \delta_0$ , where  $\delta_0$  is the Dirac delta distribution at the origin. This solution, denoted by  $\Phi$ , is called the **fundamental solution** of the Laplacian. It represents the potential generated by a unit point charge at the origin.

Assuming the solution is radially symmetric, i.e.,  $\Phi(x) = \phi(|x|)$ , we can solve the equation. For  $x \neq 0$ , we have  $\Delta \Phi = 0$ . The radial form of the Laplacian gives an ODE for  $\phi(r)$ . Solving this ODE yields:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3 \end{cases} \quad (2.15)$$

where  $\alpha(n)$  is the volume of the unit ball in  $\mathbb{R}^n$ . Note that the fundamental solution is singular at the origin and is not in  $L^1_{loc}(\mathbb{R}^n)$ .

Using this fundamental solution, we can derive a representation formula for any smooth function  $u$ .

**Theorem 3.1** (Representation Formula). *Let  $u \in C^2(\overline{\Omega})$ . Then for any  $x \in \Omega$ :*

$$u(x) = \int_{\partial\Omega} \left( u(y) \frac{\partial \Phi(y-x)}{\partial \nu(y)} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu(y)} \right) dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y) dy \quad (2.16)$$

*This formula is derived from Green's second identity.*

This formula shows that the value of  $u$  at any point  $x$  in the interior of  $\Omega$  is determined by the values of  $u$  and its normal derivative on the boundary, and the values of  $\Delta u$  in the interior.

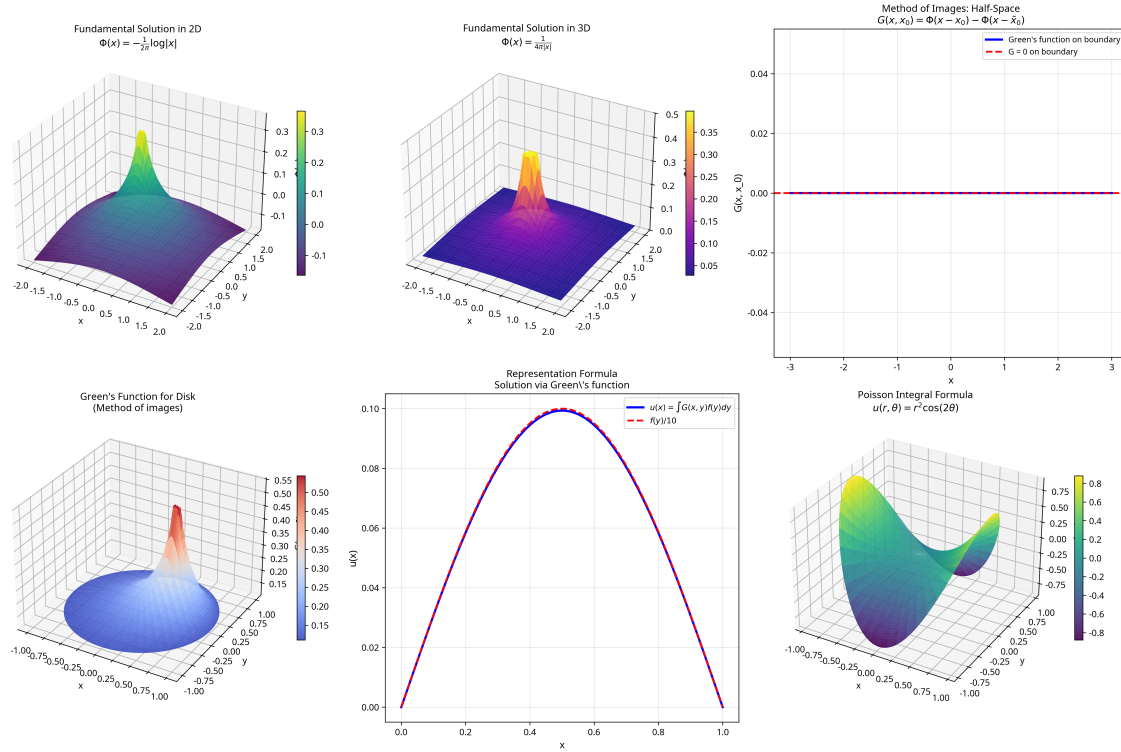


Figure 3: **Fundamental Solutions and Green's Functions.** This figure illustrates the construction and application of Green's functions. **(Top left)** The fundamental solution in 2D,  $\Phi(x) = -\frac{1}{2\pi} \log|x|$ , which is singular at the origin and represents the potential from a point source. **(Top middle)** The fundamental solution in 3D,  $\Phi(x) = \frac{1}{4\pi|x|}$ , the Coulomb potential. **(Top right)** The method of images for a half-space: the Green's function is constructed by placing an image charge of opposite sign at the reflected point, ensuring the potential vanishes on the boundary. **(Bottom left)** The Green's function for a disk, constructed using the method of images with inversion. **(Bottom middle)** Illustration of the representation formula: the solution  $u(x)$  is expressed as an integral involving the Green's function and the source term  $f$ . **(Bottom right)** The Poisson integral formula for a disk, giving the solution to Laplace's equation with prescribed boundary data.

### 3.2 Green's Function for a Domain

The representation formula is not yet a solution to the Dirichlet problem, because it involves both  $u$  and its normal derivative on the boundary. For the Dirichlet problem, only  $u$  is prescribed on the boundary. The idea of the Green's function is to correct the fundamental solution so that the term involving the unknown normal derivative vanishes.

**Definition 3.2** (Green's Function). For a domain  $\Omega$ , the **Green's function**  $G(x, y)$  is a function of two variables  $x, y \in \Omega$  defined for each fixed  $x \in \Omega$  as:

$$G(x, y) = \Phi(y - x) - \phi^x(y) \quad (2.17)$$

where the *corrector function*  $\phi^x(y)$  is a harmonic function of  $y$  in  $\Omega$  that solves the boundary value problem:

$$\begin{cases} \Delta_y \phi^x = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y - x) & \text{for } y \in \partial\Omega \end{cases} \quad (2.18)$$

By construction, for a fixed  $x$ , the Green's function  $G(x, y)$  satisfies:

1.  $G(x, y)$  is harmonic in  $y$  for  $y \neq x$ .
2.  $G(x, y) = 0$  for  $y \in \partial\Omega$ .
3.  $G(x, y)$  has the same singularity as  $\Phi(y - x)$  at  $y = x$ .

With the Green's function, we can now derive a solution formula for the Poisson equation with homogeneous Dirichlet boundary conditions.

**Theorem 3.3** (Solution using Green's Function). *If a Green's function  $G(x, y)$  exists for the domain  $\Omega$ , then the solution to the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (2.19)$$

is given by the formula:

$$u(x) = \int_{\Omega} G(x, y) f(y) dy - \int_{\partial\Omega} g(y) \frac{\partial G(x, y)}{\partial \nu(y)} dS(y) \quad (2.20)$$

This formula is a remarkable achievement. It provides an explicit solution to the boundary value problem, provided that we can find the Green's function for the domain. The problem of solving the PDE is thus reduced to the problem of finding the Green's function.

### 3.3 The Method of Images

For simple domains with high degrees of symmetry, such as a half-space or a ball, the Green's function can be constructed explicitly using the **method of images**. The idea is to place a fictitious 'image charge' outside the domain in such a way that the potential on the boundary is zero.

#### Example: Green's function for a half-space

Let  $\Omega = \mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ . The boundary is the hyperplane  $\partial\Omega = \{x_n = 0\}$ . For a point  $x = (x', x_n) \in \Omega$ , its reflection across the boundary is  $\tilde{x} = (x', -x_n)$ . The corrector function is chosen to be the potential of a point charge of opposite sign located at the reflected point  $\tilde{x}$ :

$$\phi^x(y) = \Phi(y - x) - \Phi(y - \tilde{x}) \quad (2.21)$$

One can verify that for  $y \in \partial\Omega$ , we have  $|y - x| = |y - \tilde{x}|$ , and so  $\phi^x(y) = 0$  on the boundary. The Green's function is then:

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x}) \quad (2.22)$$

#### Example: Green's function for a ball

For a ball  $B(0, R)$ , the reflection is an inversion with respect to the sphere. For a point  $x \in B(0, R)$ , its image point is  $\tilde{x} = \frac{R^2}{|x|^2} x$ . The Green's function for the ball is then given by:

$$G(x, y) = \Phi(y - x) - \Phi\left(\frac{|x|}{R}(y - \tilde{x})\right) \quad (2.23)$$

From this, one can derive the famous **Poisson integral formula** for the solution of Laplace's equation in a ball.



## 4 Perron's Method and the Solution of the Dirichlet Problem

The method of Green's functions is very powerful, but it relies on our ability to find the Green's function for a given domain. For general domains, this is not possible. Perron's method, developed by Oskar Perron in the 1920s, provides a way to prove the existence of a solution to the Dirichlet problem for Laplace's equation on a much wider class of domains, without explicitly constructing the solution.

The method is a beautiful example of the power of real analysis. The idea is to construct the solution as the supremum of a family of 'subharmonic' functions that lie below the desired boundary data.

### 4.1 Subharmonic and Superharmonic Functions

We first need to define the classes of functions that will be used to 'sandwich' the solution.

**Definition 4.1** (Subharmonic and Superharmonic Functions). Let  $\Omega \subset \mathbb{R}^n$  be an open set. A function  $u : \Omega \rightarrow [-\infty, \infty)$  is **subharmonic** if:

- (i)  $u$  is upper semi-continuous.
- (ii) For every ball  $\overline{B(x, r)} \subset \Omega$ ,  $u$  satisfies the sub-mean value property:

$$u(x) \leq \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS(y) \quad (2.24)$$

A function  $v$  is **superharmonic** if  $-v$  is subharmonic.

For smooth functions, being subharmonic is equivalent to having  $\Delta u \geq 0$ . Subharmonic functions can be thought of as functions that are 'more concave' than harmonic functions.

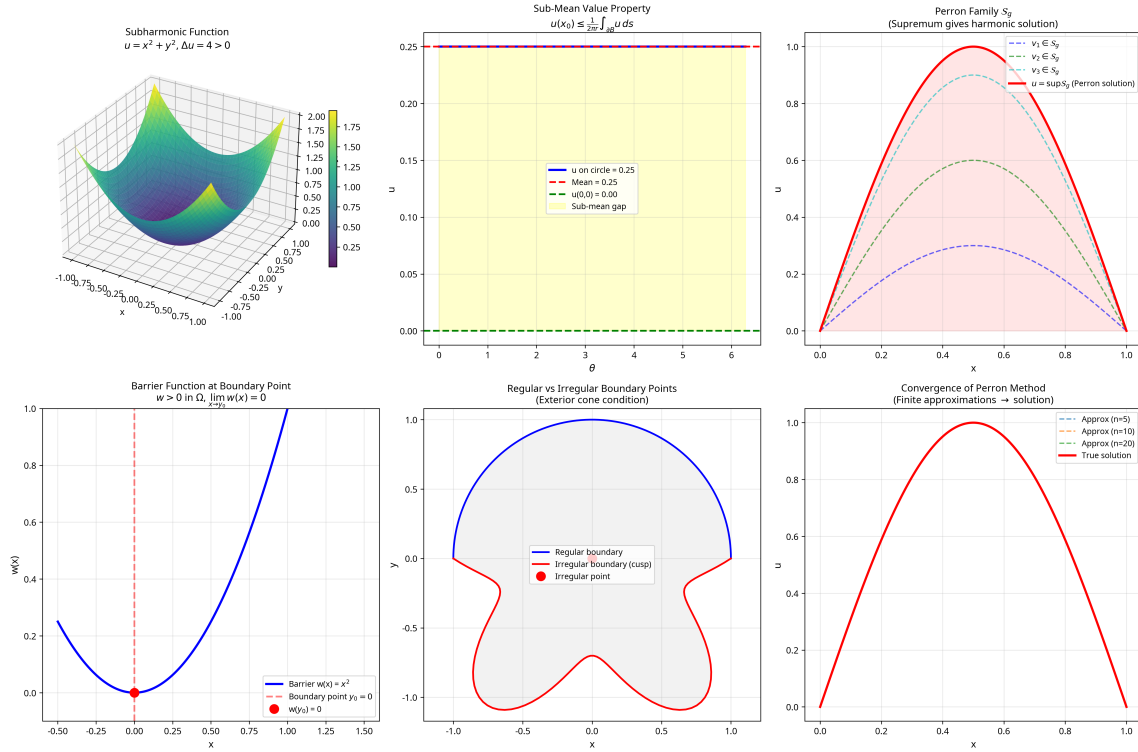


Figure 4: **Perron's Method and Subharmonic Functions.** This figure illustrates the key concepts of Perron's method. **(Top left)** An example of a subharmonic function  $u = x^2 + y^2$ , which satisfies  $\Delta u = 4 > 0$ . **(Top middle)** The sub-mean value property: the value at the centre is less than or equal to the average on the circle. **(Top right)** The Perron family  $\mathcal{S}_g$  consists of all subharmonic functions dominated by the boundary data  $g$ . The Perron solution is the supremum of this family. **(Bottom left)** A barrier function at a boundary point: a superharmonic function that is positive in the domain and vanishes at the boundary point. **(Bottom middle)** Regular vs irregular boundary points: points satisfying an exterior cone condition are regular. **(Bottom right)** Convergence of the Perron method: taking the supremum over larger subfamilies approximates the true solution.

## 4.2 The Perron Family and the Perron Solution

Now, let's consider the Dirichlet problem for Laplace's equation on a bounded domain  $\Omega$  with boundary data  $g \in C(\partial\Omega)$ .

We define the **Perron family** of subharmonic functions, denoted  $\mathcal{S}_g$ , as the set of all subharmonic functions  $v$  in  $\Omega$  such that:

$$\limsup_{x \rightarrow y} v(x) \leq g(y) \quad \text{for all } y \in \partial\Omega \quad (2.25)$$

This condition means that the functions in the Perron family are 'dominated' by the boundary data  $g$ . The family  $\mathcal{S}_g$  is non-empty, since the constant function  $v(x) = \min_{\partial\Omega} g$  is in it.

We can now define the Perron solution.

**Definition 4.2 (Perron Solution).** The **Perron solution**  $u$  to the Dirichlet problem is defined as the pointwise supremum of the functions in the Perron family:

$$u(x) := \sup\{v(x) \mid v \in \mathcal{S}_g\} \quad \text{for } x \in \Omega \quad (2.26)$$

The main result of Perron's method is that this function  $u$  is, under certain conditions on the domain, the harmonic function we are looking for.

**Theorem 4.3** (Perron's Theorem). *Let  $\Omega$  be a bounded open set. The Perron solution  $u$  defined above is harmonic in  $\Omega$ .*

*Proof Sketch.* The proof is intricate and relies on several key lemmas.

1. The supremum of a family of subharmonic functions is not necessarily subharmonic. However, one can show that the upper semi-continuous regularization of the supremum is subharmonic.
2. Let  $v_1, v_2 \in \mathcal{S}_g$ . Then  $\max(v_1, v_2)$  is also subharmonic and in  $\mathcal{S}_g$ . This shows that the Perron family is 'directed upwards'.
3. For any point  $x \in \Omega$  and any ball  $B$  around  $x$ , one can construct a 'lifting' of any function  $v \in \mathcal{S}_g$  to a new function  $V$  which is also in  $\mathcal{S}_g$ , is equal to  $v$  outside the ball, and is harmonic inside the ball. This lifting process increases the value of the function inside the ball.
4. Using this lifting property, one can show that the Perron solution  $u$  must satisfy the mean value property, and is therefore harmonic.

□

### 4.3 Barrier Functions and Regular Points

Perron's theorem tells us that the function  $u$  we constructed is harmonic. However, it does not guarantee that  $u$  actually attains the prescribed boundary values  $g$ . That is, we do not yet know if  $\lim_{x \rightarrow y} u(x) = g(y)$  for  $y \in \partial\Omega$ .

This is where the concept of a **barrier function** comes in. A barrier function is a function that 'traps' the solution at a boundary point, forcing it to take on the correct value.

**Definition 4.4** (Barrier). A point  $y_0 \in \partial\Omega$  is called a **regular point** for the Dirichlet problem if for every continuous function  $g$  on  $\partial\Omega$ , the Perron solution  $u$  satisfies  $\lim_{x \rightarrow y_0} u(x) = g(y_0)$ . A point is regular if there exists a **barrier** at that point. A barrier at  $y_0$  is a superharmonic function  $w$  defined on  $\Omega$  such that  $w > 0$  on  $\Omega$  and  $\lim_{x \rightarrow y_0} w(x) = 0$ .

The existence of a barrier at every point on the boundary is a purely geometric condition on the domain  $\Omega$ . For example, if the boundary of  $\Omega$  satisfies an **exterior cone condition** at every point (meaning that at every boundary point, one can place a small cone that lies entirely outside  $\Omega$ ), then every point on the boundary is regular. This condition is satisfied by all domains with  $C^1$  boundaries, and many other domains with corners.

If every point on the boundary of  $\Omega$  is regular, then Perron's method provides a complete solution to the classical Dirichlet problem for Laplace's equation.

## 5 Regularity of Solutions

A central theme in the theory of elliptic equations is that of **elliptic regularity**. This refers to the remarkable property that weak solutions to elliptic equations are often much smoother than one would expect from the data. For instance, if the source term  $f$  in the Poisson equation  $-\Delta u = f$  is smooth, then the solution  $u$  is also smooth.

### 5.1 Interior Regularity

We have already seen a hint of this in the fact that any continuous function satisfying the mean value property is automatically  $C^\infty$ . This is a general feature of harmonic functions.

**Theorem 5.1** (Interior Regularity for Laplace's Equation). *If  $u \in L^1_{loc}(\Omega)$  is a weak solution to Laplace's equation (i.e.,  $\int u \Delta \phi = 0$  for all  $\phi \in C_c^\infty(\Omega)$ ), then  $u$  is equal almost everywhere to a function that is  $C^\infty(\Omega)$  and harmonic in the classical sense.*

This is a powerful result. It says that for Laplace's equation, the concept of a weak solution is not really more general than that of a classical solution in the interior of the domain. The proof involves convolving the weak solution with a family of smooth mollifiers and showing that the resulting smooth functions converge to a smooth solution.

For the general elliptic equation  $Lu = f$ , the regularity of the solution depends on the regularity of the coefficients of the operator  $L$  and the source term  $f$ .

**Theorem 5.2** (Interior Regularity for General Elliptic Equations). *Let  $L$  be a uniformly elliptic operator with coefficients  $a_{ij}, b_i, c$  that are  $C^\infty(\Omega)$ . If  $u \in H^1(\Omega)$  is a weak solution to  $Lu = f$  and  $f \in C^\infty(\Omega)$ , then  $u \in C^\infty(\Omega)$ .*

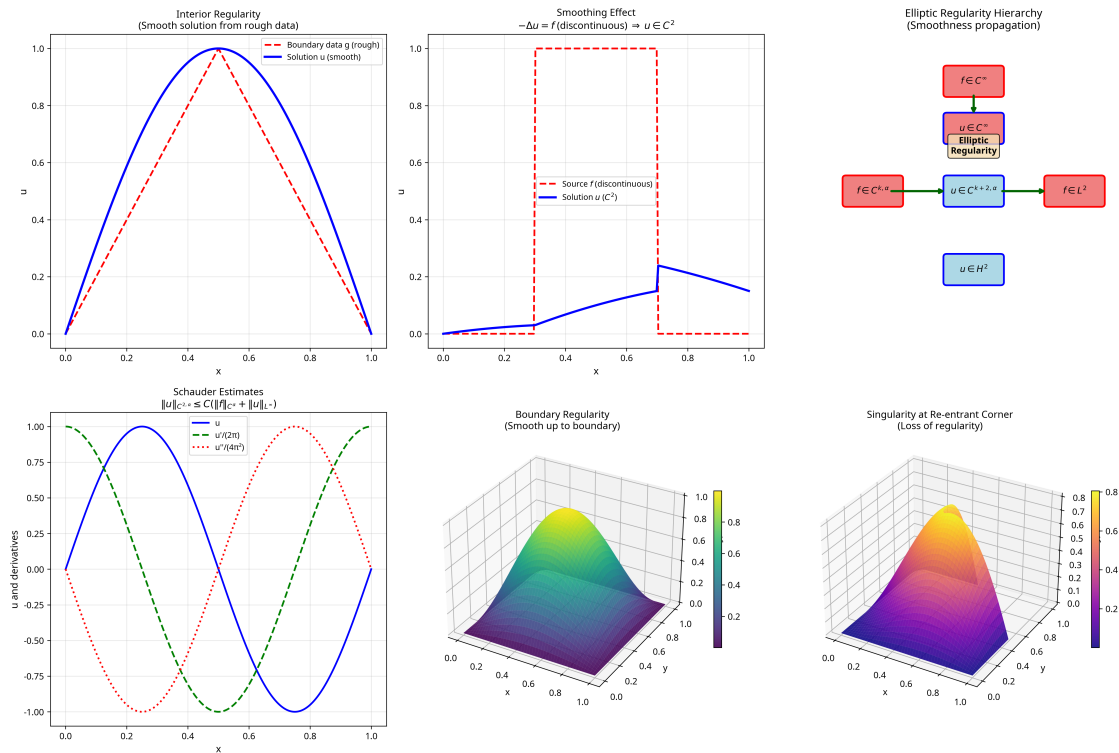


Figure 5: **Elliptic Regularity and Smoothness of Solutions.** This figure demonstrates the regularity properties of elliptic equations. **(Top left)** Interior regularity: even with rough boundary data, the solution is smooth in the interior. **(Top middle)** The smoothing effect of elliptic operators: a discontinuous source term  $f$  produces a  $C^2$  solution  $u$ . **(Top right)** The elliptic regularity hierarchy: the smoothness of the source term  $f$  determines the smoothness of the solution  $u$ . If  $f \in C^{k,\alpha}$ , then  $u \in C^{k+2,\alpha}$ . **(Bottom left)** Illustration of Schauder estimates: the norms of the solution and its derivatives are controlled by the norm of the source term. **(Bottom middle)** Boundary regularity: when the boundary and data are smooth, the solution is smooth up to the boundary. **(Bottom right)** Loss of regularity at a re-entrant corner: the solution develops a singularity at corners with angles greater than  $\pi$ .

## 5.2 Regularity up to the Boundary

Obtaining regularity of the solution up to the boundary is a more delicate matter, as it depends on the smoothness of the boundary itself. The theory for this is much more involved and culminates in the **Schauder estimates**.

The Schauder estimates provide a priori bounds for the solutions of elliptic equations in Hölder spaces. For an elliptic operator  $L$  with Hölder continuous coefficients and a source term  $f$  that is also Hölder continuous, the Schauder interior estimates state that the  $C^{2,\alpha}$  norm of the solution  $u$  in any compact subdomain is bounded by the  $C^\alpha$  norm of  $f$  and the  $L^\infty$  norm of  $u$ .

**Theorem 5.3** (Schauder Interior Estimates). *Let  $u \in C^{2,\alpha}(\Omega)$  be a solution of  $Lu = f$  in  $\Omega$ . Then for any subdomain  $\Omega' \Subset \Omega$ , there is a constant  $C$  such that*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|f\|_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)}) \quad (2.27)$$

There are also Schauder estimates up to the boundary, which require the boundary and the boundary data to be sufficiently smooth. These estimates are a key tool for proving the existence of classical solutions to the Dirichlet problem for general linear elliptic equations using the **method of continuity**.

## 6 Applications of Elliptic Equations

Elliptic equations arise naturally in a wide variety of physical and geometric contexts. We briefly survey some of the most important applications.

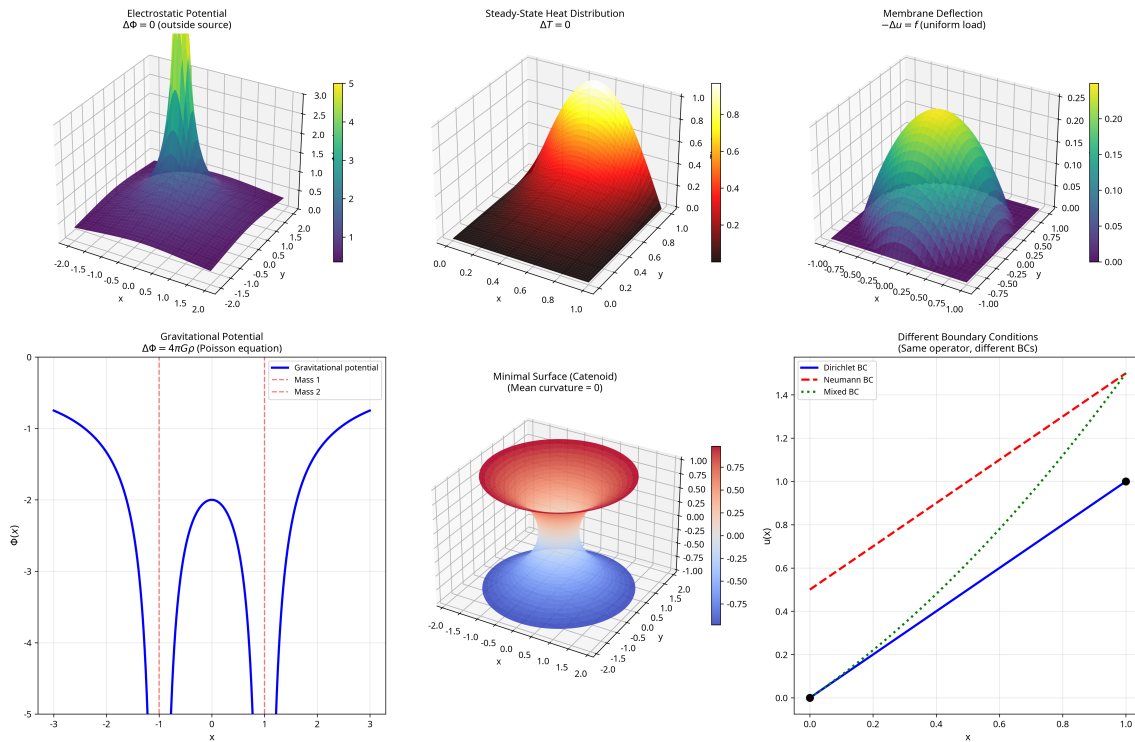


Figure 6: **Applications of Elliptic Equations.** This figure illustrates various physical and geometric applications of elliptic PDEs. **(Top left)** Electrostatic potential from a point charge, satisfying Laplace's equation  $\Delta\Phi = 0$  outside the source. **(Top middle)** Steady-state heat distribution in a plate with prescribed boundary temperatures, governed by  $\Delta T = 0$ . **(Top right)** Deflection of a circular membrane under uniform load, modelled by the Poisson equation  $-\Delta u = f$ . **(Bottom left)** Gravitational potential from two point masses, illustrating the Poisson equation  $\Delta\Phi = 4\pi G\rho$ . **(Bottom middle)** A minimal surface (catenoid), which has zero mean curvature and is governed by a nonlinear elliptic equation. **(Bottom right)** Comparison of solutions with different boundary conditions (Dirichlet, Neumann, mixed) for the same operator.



**Electrostatics.** The electrostatic potential  $\Phi$  in a region free of charges satisfies Laplace's equation  $\Delta\Phi = 0$ . In the presence of a charge distribution  $\rho$ , the potential satisfies the Poisson equation  $-\Delta\Phi = \rho/\epsilon_0$ .

**Steady-state heat conduction.** The temperature distribution  $T$  in a body in thermal equilibrium satisfies  $\Delta T = 0$  in regions without heat sources. With a heat source of density  $f$ , we have  $-\Delta T = f/k$ , where  $k$  is the thermal conductivity.

**Membrane deflection.** The vertical deflection  $u$  of an elastic membrane under a transverse load  $f$  is governed by  $-\Delta u = f/\tau$ , where  $\tau$  is the tension.

**Minimal surfaces.** A surface that minimizes area subject to boundary constraints has zero mean curvature and is described by a nonlinear elliptic equation.

## 7 Conclusion

This chapter has provided an overview of the classical theory of second-order linear elliptic equations. We have seen that harmonic functions possess remarkable properties, such as the mean value property and the maximum principle, which have profound consequences for the uniqueness and stability of solutions. We developed the method of Green's functions, which provides an explicit integral representation of the solution, and Perron's method, which gives a general existence proof for the Dirichlet problem. Finally, we discussed the fundamental concept of elliptic regularity, which ensures that solutions to elliptic equations are as smooth as the data allows. This classical theory provides the foundation and intuition for the more abstract, functional-analytic approach to elliptic PDEs that was developed in the previous chapter, and prepares us for the study of parabolic and hyperbolic equations in the chapters to come.

## References

- [1] Gilbarg, D., & Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order* (2nd ed.). Springer.
- [2] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.
- [3] Perron, O. (1923). Eine neue Behandlung der ersten Randwertaufgabe für  $\Delta u = 0$ . *Mathematische Zeitschrift*, 18(1), 42–54.
- [4] Schauder, J. (1934). Über lineare elliptische Differentialgleichungen zweiter Ordnung. *Mathematische Zeitschrift*, 38(1), 257–282.
- [5] Courant, R., & Hilbert, D. (1953). *Methods of Mathematical Physics, Volume II: Partial Differential Equations*. Interscience Publishers.

## Chapter 3

# Parabolic Equations: The Heat Equation and Evolution Problems

## 1 Introduction to Parabolic Equations and the Heat Equation

We now transition from the study of steady-state phenomena, described by elliptic equations, to the analysis of time-dependent evolution processes. The archetypal model for such processes is the **heat equation**, which governs the diffusion of heat in a medium over time. This equation serves as the prototype for a broad class of partial differential equations known as **parabolic equations**.

Parabolic PDEs are fundamentally different from their elliptic counterparts. They describe problems that are second-order in space but only first-order in time. This temporal asymmetry introduces an irreversible “arrow of time” into the mathematics. Unlike elliptic equations, which have smooth solutions globally, parabolic equations exhibit an infinite speed of propagation for disturbances, meaning a change at any point in the initial data is felt instantly, albeit infinitesimally, everywhere in the domain. However, they also possess a strong regularising effect, smoothing out initial data as time progresses.

A general second-order linear parabolic operator  $L$  in a domain  $\Omega \subset \mathbb{R}^n$  and for time  $t > 0$  is given by:

$$Lu = u_t - \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u \quad (3.1)$$

The operator is parabolic if the spatial part is elliptic, i.e., the matrix  $A(x,t) = (a_{ij}(x,t))$  is positive definite for all  $(x,t)$ . The quintessential parabolic equation is the heat equation, where  $A$  is the identity matrix and the lower-order terms are zero:

$$u_t - \Delta u = 0 \quad (3.2)$$

This chapter will develop the classical theory for parabolic equations, focusing primarily on the heat equation. We will derive the equation from physical principles, introduce the concept of the initial/boundary value problem, and explore the fundamental solution. We will then prove a maximum principle for the heat equation, which, like its elliptic counterpart, is a cornerstone of the theory and provides uniqueness for solutions. Finally, we will discuss the regularity of solutions, highlighting the smoothing property of the heat operator.

### 1.1 Derivation of the Heat Equation

The heat equation can be derived from two fundamental physical principles: the conservation of energy and Fourier’s law of heat conduction.

Consider a region  $V \subset \mathbb{R}^n$ . The total amount of heat energy contained in this region at time  $t$  is given by the integral of the energy density:

$$E(t) = \int_V c\rho u(x, t) dx \quad (3.3)$$

where  $u(x, t)$  is the temperature at point  $x$  and time  $t$ ,  $c$  is the specific heat capacity of the material, and  $\rho$  is its density. The rate of change of this energy must be equal to the heat flowing across the boundary  $\partial V$  plus the heat generated by any sources inside  $V$ .

The **conservation of energy** principle states:

$$\frac{dE}{dt} = - \int_{\partial V} \mathbf{q} \cdot \mathbf{n} dS + \int_V f(x, t) dx \quad (3.4)$$

where  $\mathbf{q}$  is the heat flux vector (energy per unit area per unit time),  $\mathbf{n}$  is the outward unit normal to the boundary  $\partial V$ , and  $f(x, t)$  is the rate of heat production per unit volume from internal sources.

By the Divergence Theorem, the boundary integral can be converted to a volume integral:

$$\int_{\partial V} \mathbf{q} \cdot \mathbf{n} dS = \int_V \operatorname{div}(\mathbf{q}) dx \quad (3.5)$$

Substituting this and the expression for  $E(t)$  into the energy balance equation, we get:

$$\int_V c\rho u_t(x, t) dx = - \int_V \operatorname{div}(\mathbf{q}) dx + \int_V f(x, t) dx \quad (3.6)$$

Since this must hold for any arbitrary volume  $V$ , the integrands must be equal:

$$c\rho u_t = -\operatorname{div}(\mathbf{q}) + f \quad (3.7)$$

Now, we need a constitutive relation that connects the heat flux  $\mathbf{q}$  to the temperature  $u$ . This is provided by **Fourier's law of heat conduction**, which states that heat flows from hotter regions to colder regions, and the rate of flow is proportional to the temperature gradient:

$$\mathbf{q} = -k\nabla u \quad (3.8)$$

where  $k > 0$  is the thermal conductivity of the material.

Substituting Fourier's law into the conservation equation, we obtain:

$$c\rho u_t = -\operatorname{div}(-k\nabla u) + f = \operatorname{div}(k\nabla u) + f \quad (3.9)$$

If the material is homogeneous, then  $k$  is constant, and we can write:

$$c\rho u_t = k\Delta u + f \quad (3.10)$$

Rearranging this gives the inhomogeneous heat equation:

$$u_t - D\Delta u = \frac{f}{c\rho} \quad (3.11)$$

where  $D = k/(c\rho)$  is the thermal diffusivity. In the absence of sources ( $f = 0$ ) and setting  $D = 1$  by scaling time or space, we arrive at the classical heat equation:

$$u_t - \Delta u = 0 \quad (3.12)$$

This derivation highlights that the heat equation is a mathematical formulation of the fundamental principles of energy conservation and heat flow.

## 2 The Initial/Boundary Value Problem

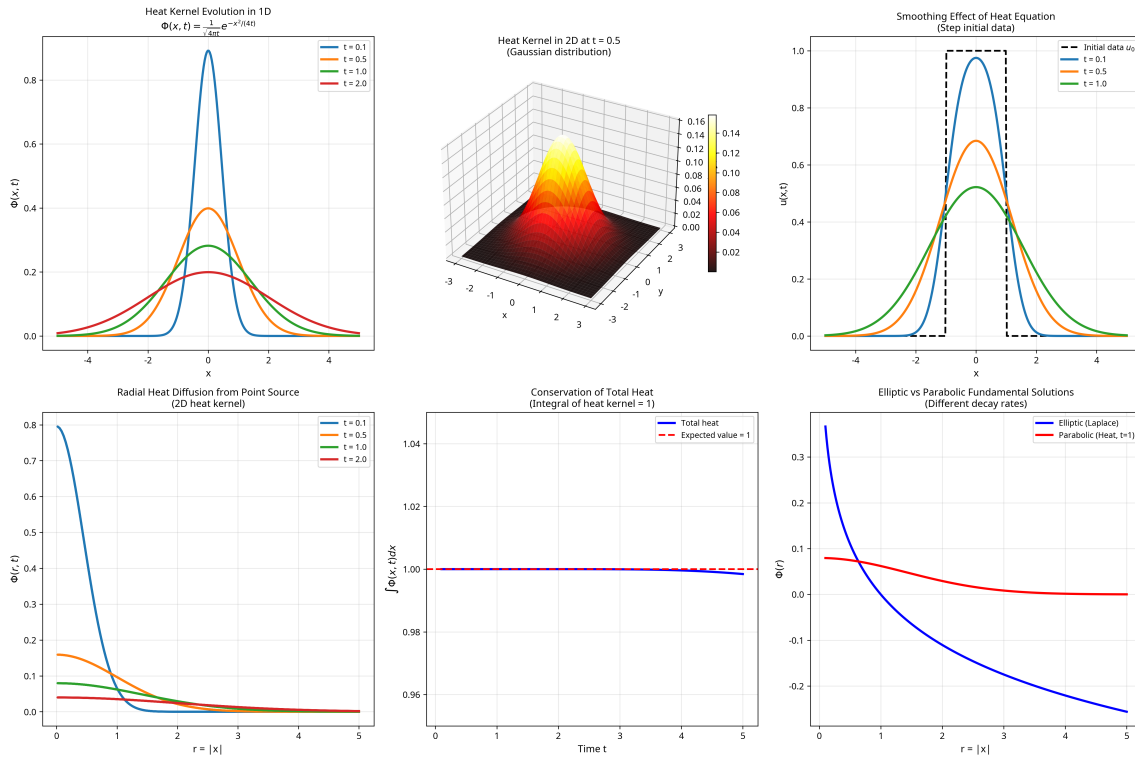
Unlike elliptic problems, which are typically posed as pure boundary value problems, parabolic equations describe evolution in time and thus require initial conditions to be specified. A well-posed problem for the heat equation typically involves specifying the initial temperature distribution throughout the domain and the temperature or heat flux on the boundary for all subsequent times.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and let  $T > 0$  be a final time. We define the space-time cylinder  $U_T = \Omega \times (0, T]$  and its parabolic boundary  $\Gamma_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T])$ .

The initial/boundary value problem for the heat equation is to find a function  $u(x, t)$  such that:

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \partial\Omega \times (0, T] \quad (\text{Dirichlet boundary condition}) \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega \quad (\text{Initial condition}) \end{cases} \quad (3.13)$$

Here,  $f$  is a source term,  $g$  is the prescribed boundary temperature, and  $u_0$  is the initial temperature distribution. One could also prescribe Neumann boundary conditions (specifying the heat flux  $\frac{\partial u}{\partial \nu}$  on the boundary) or Robin boundary conditions.



**Figure 1: Heat Equation and Fundamental Solution.** This comprehensive figure illustrates the fundamental properties of the heat equation and its solution. **(Top left)** Evolution of the heat kernel in 1D, showing how the Gaussian spreads and flattens over time. **(Top middle)** The heat kernel in 2D at a fixed time, displaying the characteristic Gaussian profile centred at the origin. **(Top right)** The smoothing effect of the heat equation: a discontinuous step initial condition becomes smooth for any  $t > 0$ . **(Bottom left)** Radial heat diffusion from a point source in 2D, showing how the peak decreases and the spread increases over time. **(Bottom middle)** Conservation of total heat: the integral of the heat kernel remains constant at 1 for all times, reflecting energy conservation. **(Bottom right)** Comparison of elliptic and parabolic fundamental solutions, showing different decay rates.

### 3 The Fundamental Solution

As with the Laplacian, we can find a fundamental solution for the heat operator  $L = \partial_t - \Delta$ . This is a solution to the equation  $Lu = \delta_{(0,0)}$ , representing the temperature evolution from a point source of heat released at the origin at time  $t = 0$ . The fundamental solution, denoted  $\Phi(x, t)$ , is given by:

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \quad (3.14)$$

This function is also known as the **heat kernel**. It has several important properties:

1. For each fixed  $t > 0$ , the function  $x \mapsto \Phi(x, t)$  is a Gaussian function. As  $t \rightarrow 0^+$ , the Gaussian becomes increasingly peaked at the origin, approximating a Dirac delta function.
2. For any  $t > 0$ ,  $\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$ . This reflects the conservation of energy: the total amount of heat from the initial point source remains constant over time.
3. For any  $x \neq 0$  and  $t > 0$ ,  $\Phi_t - \Delta\Phi = 0$ . The fundamental solution is a classical solution of the heat equation away from the origin.

Using the fundamental solution, we can write down an explicit solution to the initial value problem for the heat equation on the whole space  $\mathbb{R}^n$  (the Cauchy problem):

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^n \end{cases} \quad (3.15)$$

The solution is given by the convolution of the initial data with the heat kernel:

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) u_0(y) dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \quad (3.16)$$

This formula shows that the temperature at a point  $(x, t)$  is a weighted average of the initial temperatures, with the weights given by the Gaussian kernel. The influence of the initial temperature at a point  $y$  on the temperature at  $x$  decays exponentially with the square of the distance  $|x - y|^2$ . However, for any  $t > 0$ , the value of  $u(x, t)$  depends on the initial data over the entire space  $\mathbb{R}^n$ . This is the mathematical manifestation of the **infinite speed of propagation** of heat.

### 4 The Maximum Principle for the Heat Equation

Just as for elliptic equations, the maximum principle is a fundamental tool in the study of parabolic equations. It states that the maximum temperature in a region of space-time must occur either at the initial time or on the spatial boundary.

Let  $U_T = \Omega \times (0, T]$  be the space-time cylinder, and let its parabolic boundary be  $\Gamma_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T])$ .

**Theorem 4.1** (Weak Maximum Principle for the Heat Equation). *Let  $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$  be a solution to the heat equation  $u_t - \Delta u = 0$  in  $U_T$ . Then the maximum value of  $u$  is attained on the parabolic boundary  $\Gamma_T$ :*

$$\max_{\overline{U_T}} u = \max_{\Gamma_T} u \quad (3.17)$$



*Proof Sketch.* The proof is slightly more subtle than for the elliptic case because we cannot simply assume the maximum occurs at an interior point and use the equation to get a contradiction. If  $u$  has a maximum at an interior point  $(x_0, t_0)$ , we would have  $\nabla u(x_0, t_0) = 0$ ,  $u_t(x_0, t_0) = 0$ , and the Hessian matrix of the spatial derivatives  $D^2u(x_0, t_0)$  would be negative semi-definite, implying  $\Delta u(x_0, t_0) \leq 0$ . Then the heat equation gives  $u_t - \Delta u = 0 - (\leq 0) \geq 0$ . This does not give a contradiction.

To circumvent this, we use a trick. Consider the auxiliary function  $v(x, t) = u(x, t) - \epsilon t$  for some small  $\epsilon > 0$ . Then  $v_t - \Delta v = (u_t - \epsilon) - \Delta u = (u_t - \Delta u) - \epsilon = -\epsilon < 0$ . Now, if  $v$  were to have a maximum at an interior point  $(x_0, t_0)$ , we would have  $v_t(x_0, t_0) = 0$  and  $\Delta v(x_0, t_0) \leq 0$ . This would imply  $v_t - \Delta v \geq 0$  at that point, which contradicts  $v_t - \Delta v = -\epsilon < 0$ . Therefore,  $v$  must attain its maximum on the parabolic boundary  $\Gamma_T$ .

Now, we have:

$$\max_{\overline{U_T}} u = \max_{\overline{U_T}} (v + \epsilon t) \leq \max_{\overline{U_T}} v + \epsilon T = \max_{\Gamma_T} v + \epsilon T \leq \max_{\Gamma_T} u + \epsilon T \quad (3.18)$$

Taking the limit as  $\epsilon \rightarrow 0$ , we get  $\max_{\overline{U_T}} u \leq \max_{\Gamma_T} u$ . Since the other inequality is trivial, the result follows.  $\square$

Similar to the elliptic case, there is also a strong maximum principle for the heat equation, which states that if a solution attains its maximum at an interior point of the space-time cylinder, it must be constant.

The maximum principle has the same important consequence of ensuring uniqueness for the initial/boundary value problem for the heat equation.

**Corollary 4.2** (Uniqueness for the Initial/Boundary Value Problem). *The initial/boundary value problem for the heat equation has at most one solution.*

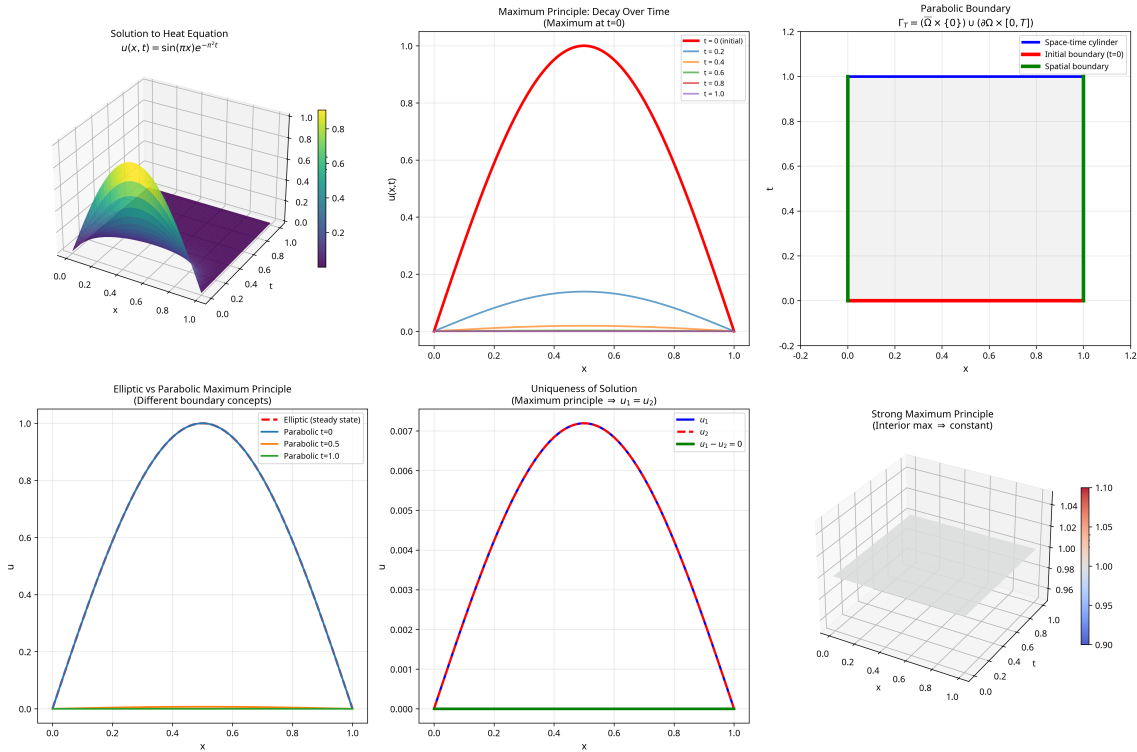


Figure 2: **Maximum Principle for the Heat Equation.** This figure demonstrates the maximum principle and its consequences for parabolic equations. **(Top left)** A solution to the heat equation on the interval  $[0, 1]$ , showing the space-time evolution. **(Top middle)** Cross-sections at different times, clearly showing that the maximum decays over time and occurs at the initial time  $t = 0$ . **(Top right)** Illustration of the parabolic boundary  $\Gamma_T$ , consisting of the initial boundary (bottom) and the spatial boundary (sides). **(Bottom left)** Comparison with the elliptic maximum principle, showing the different boundary concepts. **(Bottom middle)** Uniqueness of the solution via the maximum principle: if two solutions exist, their difference must be zero. **(Bottom right)** The strong maximum principle: if the maximum is attained at an interior point, the solution must be constant.

## 5 Regularity of Solutions

The solution formula involving the heat kernel shows that even if the initial data  $u_0$  is discontinuous, the solution  $u(x, t)$  is infinitely differentiable with respect to both space and time for any  $t > 0$ . This is a manifestation of the **smoothing property** of the heat equation.

**Theorem 5.1** (Regularity of Solutions to the Heat Equation). *Let  $u_0 \in L^1(\mathbb{R}^n)$  be the initial data. The solution  $u(x, t)$  to the Cauchy problem for the heat equation is in  $C^\infty(\mathbb{R}^n \times (0, \infty))$ .*

*Proof Sketch.* The solution is given by the convolution  $u(x, t) = (\Phi(\cdot, t) * u_0)(x)$ . Since the heat kernel  $\Phi(x, t)$  is a  $C^\infty$  function for  $t > 0$ , we can differentiate under the integral sign with respect to  $x$  and  $t$  as many times as we like. The derivatives of the Gaussian kernel are still well-behaved functions that decay rapidly, ensuring that the integrals converge. This shows that  $u(x, t)$  is infinitely differentiable for  $t > 0$ .  $\square$

This smoothing property is a hallmark of parabolic equations and stands in sharp contrast to hyperbolic equations (like the wave equation), which can propagate singularities from the initial data.

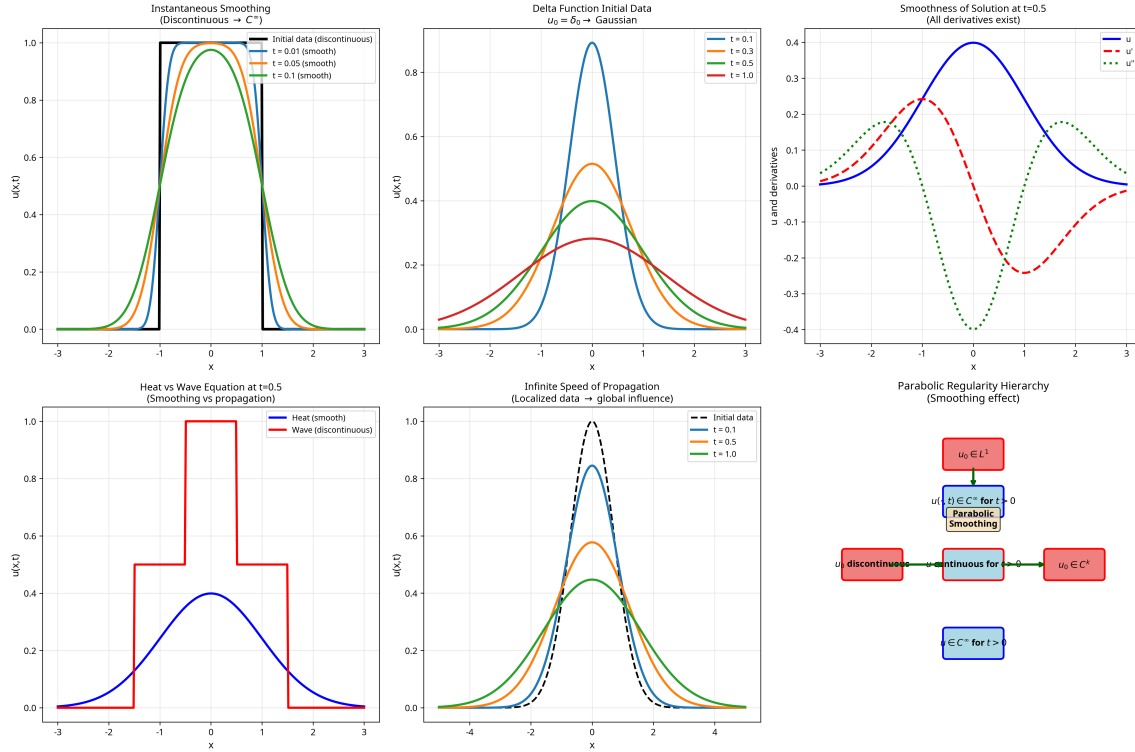


Figure 3: **Smoothing Property and Regularity.** This figure illustrates the remarkable smoothing effect of the heat equation. **(Top left)** Instantaneous smoothing: a discontinuous initial condition becomes infinitely differentiable for any  $t > 0$ . **(Top middle)** Evolution from a delta function initial data to a smooth Gaussian. **(Top right)** All derivatives of the solution exist and are continuous for  $t > 0$ . **(Bottom left)** Comparison with the wave equation: the heat equation smooths while the wave equation propagates discontinuities. **(Bottom middle)** Infinite speed of propagation: localized initial data has global influence for any  $t > 0$ . **(Bottom right)** The parabolic regularity hierarchy: the smoothness of the solution improves dramatically for  $t > 0$ , regardless of the regularity of the initial data.

## 6 Separation of Variables and Eigenfunction Expansion

For the heat equation on a bounded domain with homogeneous boundary conditions, the method of separation of variables provides an explicit solution in the form of an eigenfunction expansion. This method reveals the spectral structure of the heat operator and shows how different spatial modes decay at different rates.

Consider the heat equation on the interval  $(0, 1)$  with Dirichlet boundary conditions:

$$\begin{cases} u_t - u_{xx} = 0 & \text{for } x \in (0, 1), t > 0 \\ u(0, t) = u(1, t) = 0 & \text{for } t > 0 \\ u(x, 0) = u_0(x) & \text{for } x \in (0, 1) \end{cases} \quad (3.19)$$

We seek solutions of the form  $u(x, t) = X(x)T(t)$ . Substituting into the heat equation and separating variables gives:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad (3.20)$$

where  $\lambda$  is a separation constant. The spatial part gives the eigenvalue problem:

$$\begin{cases} -X'' = \lambda X & \text{for } x \in (0, 1) \\ X(0) = X(1) = 0 \end{cases} \quad (3.21)$$

The eigenfunctions and eigenvalues are:

$$X_n(x) = \sin(n\pi x), \quad \lambda_n = (n\pi)^2, \quad n = 1, 2, 3, \dots \quad (3.22)$$

The temporal part gives:

$$T_n(t) = e^{-\lambda_n t} = e^{-(n\pi)^2 t} \quad (3.23)$$

The general solution is a superposition of these modes:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-(n\pi)^2 t} \quad (3.24)$$

where the coefficients  $c_n$  are determined by the initial condition:

$$c_n = 2 \int_0^1 u_0(x) \sin(n\pi x) dx \quad (3.25)$$

This expansion shows that higher spatial frequencies (larger  $n$ ) decay exponentially faster, as they have larger eigenvalues. This is the mathematical reason for the smoothing property of the heat equation.

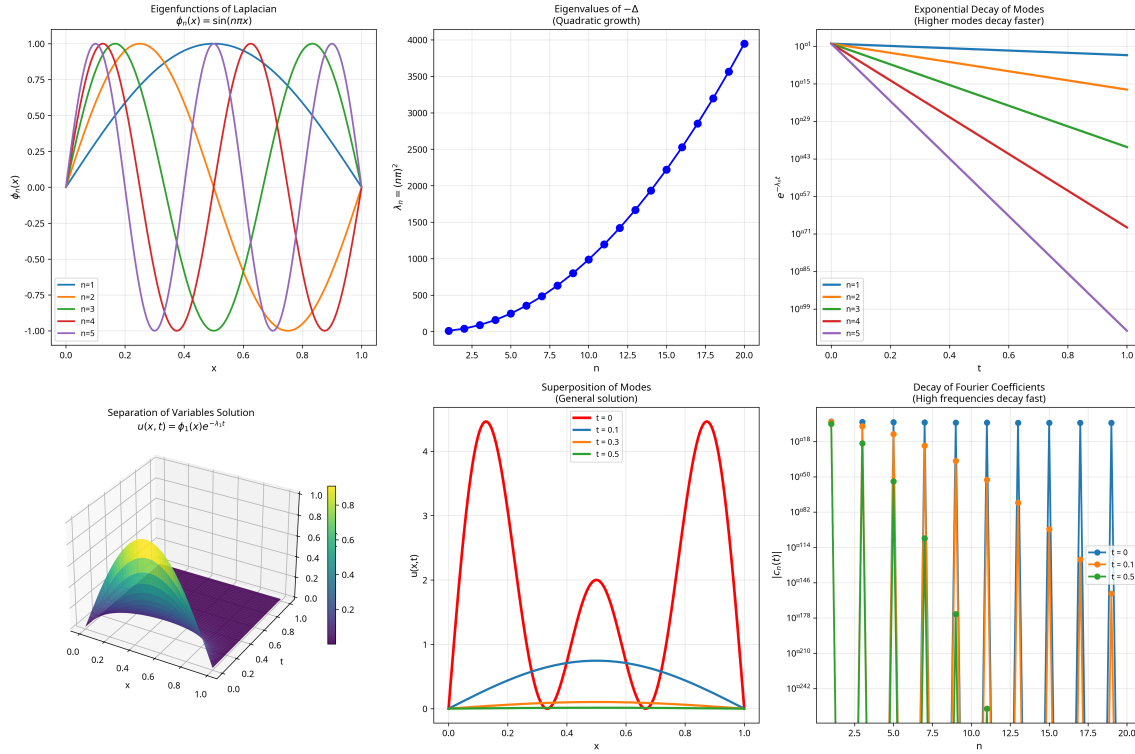


Figure 4: **Separation of Variables and Eigenfunction Expansion.** This figure illustrates the spectral decomposition of solutions to the heat equation. **(Top left)** The first five eigenfunctions of the Laplacian on  $[0, 1]$  with Dirichlet boundary conditions. **(Top middle)** The eigenvalues grow quadratically with the mode number  $n$ . **(Top right)** Exponential decay of different modes over time, with higher modes decaying faster. **(Bottom left)** A solution obtained by separation of variables, showing the space-time evolution of a single mode. **(Bottom middle)** Superposition of multiple modes to construct a general solution. **(Bottom right)** Decay of Fourier coefficients over time, demonstrating how high-frequency components are rapidly eliminated.

## 7 Energy Methods and Semigroup Theory

An alternative approach to studying the heat equation is through energy methods and the theory of semigroups. The energy of a solution is defined as:

$$E(t) = \frac{1}{2} \int_{\Omega} u^2(x, t) dx = \frac{1}{2} \|u(\cdot, t)\|_{L^2(\Omega)}^2 \quad (3.26)$$

For solutions to the heat equation with homogeneous Dirichlet boundary conditions, the energy decays monotonically in time. Differentiating with respect to time and using the heat equation:

$$\frac{dE}{dt} = \int_{\Omega} uu_t dx = \int_{\Omega} u \Delta u dx = - \int_{\Omega} |\nabla u|^2 dx \leq 0 \quad (3.27)$$

where we have used integration by parts and the boundary conditions. This shows that energy is dissipated, and the rate of dissipation is given by the  $L^2$  norm of the gradient.

The heat equation can also be viewed as an evolution equation in an abstract Hilbert space. Define the heat semigroup  $S(t) : L^2(\Omega) \rightarrow L^2(\Omega)$  by  $S(t)u_0 = u(\cdot, t)$ , where  $u$  is the solution to the heat equation with initial data  $u_0$ . This semigroup has the properties:

1.  $S(0) = I$  (identity)

2.  $S(t + s) = S(t) \circ S(s)$  (semigroup property)
3.  $\|S(t)u_0\|_{L^2} \leq \|u_0\|_{L^2}$  (contraction property)

The infinitesimal generator of the semigroup is the Laplacian operator  $A = \Delta$  with appropriate boundary conditions.

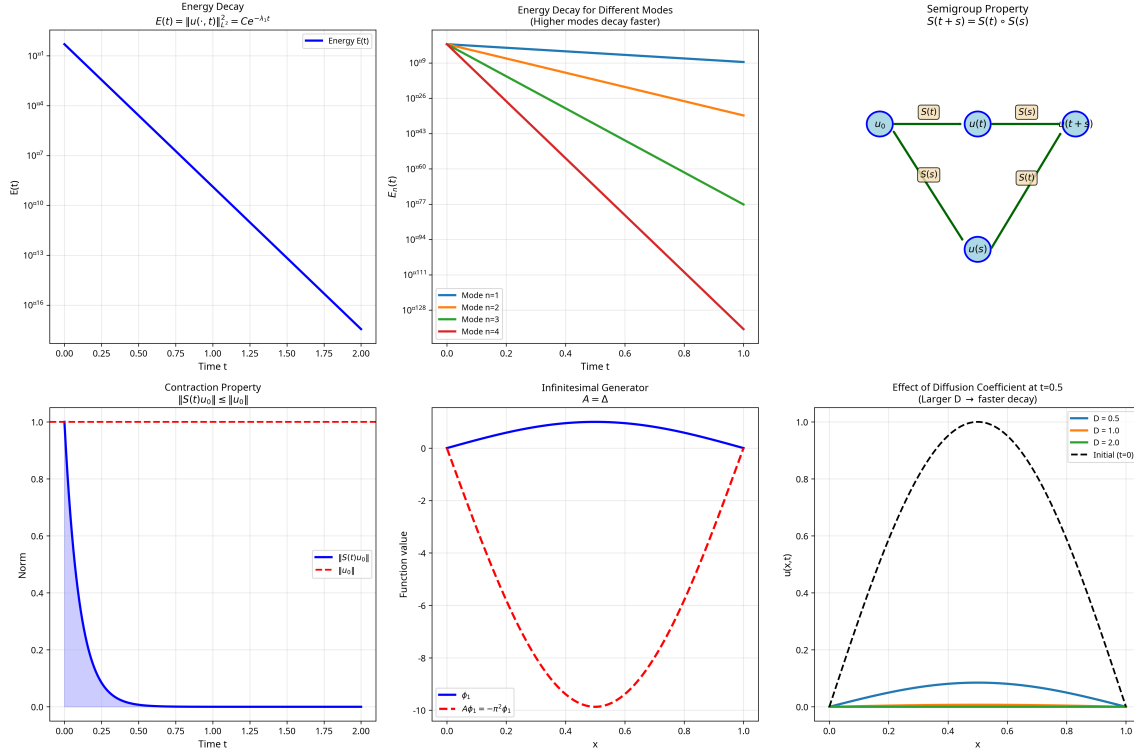


Figure 5: **Energy Methods and Semigroup Theory.** This figure illustrates the energy decay and semigroup structure of the heat equation. **(Top left)** Exponential decay of energy over time for a solution to the heat equation. **(Top middle)** Comparison of energy decay rates for different modes, showing that higher modes lose energy faster. **(Top right)** Illustration of the semigroup property:  $S(t + s) = S(t) \circ S(s)$ . **(Bottom left)** The contraction property: the norm of the solution decreases over time. **(Bottom middle)** The infinitesimal generator  $A = \Delta$  and its action on eigenfunctions. **(Bottom right)** Effect of different diffusion coefficients on the decay rate.

## 8 Applications of the Heat Equation

The heat equation and its generalizations arise in numerous physical, biological, and financial contexts. We briefly survey some of the most important applications.

**Heat diffusion.** The most direct application is to the conduction of heat in solids, liquids, and gases. The temperature distribution in a conducting medium evolves according to the heat equation.

**Diffusion processes.** The heat equation also describes the diffusion of particles, chemicals, or populations. In this context,  $u(x, t)$  represents concentration rather than temperature.

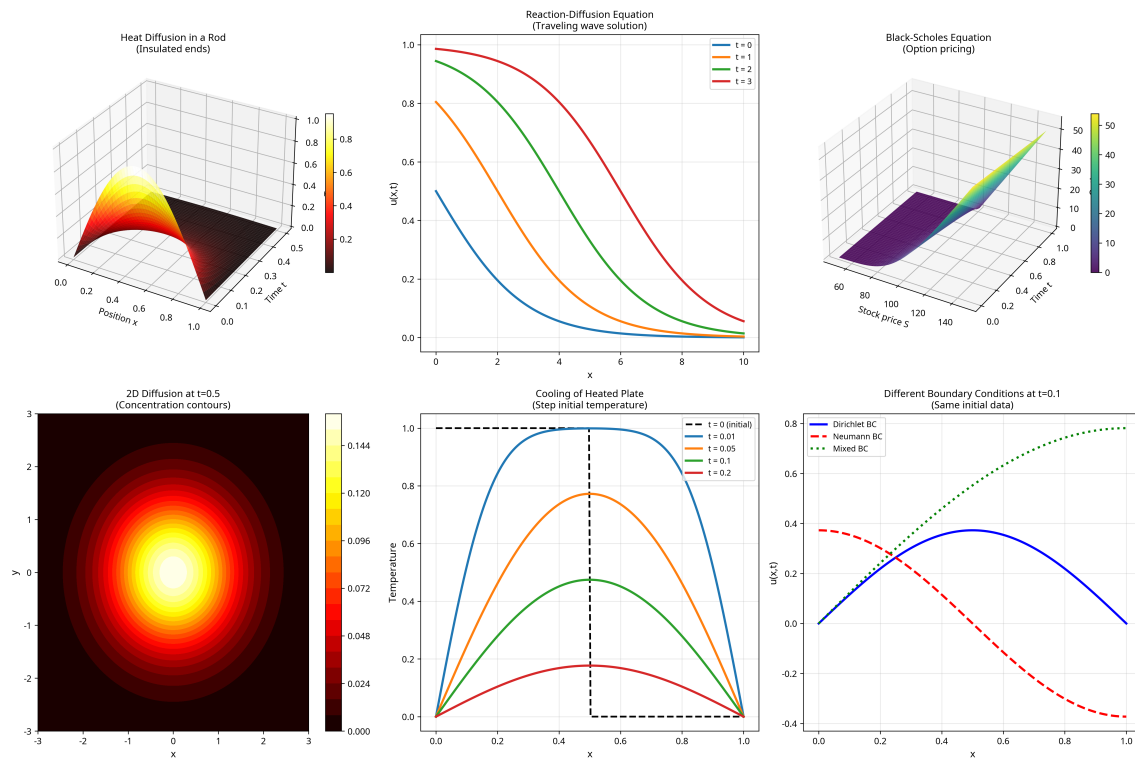
**Reaction-diffusion equations.** Adding a nonlinear reaction term to the heat equation gives reaction-diffusion equations of the form  $u_t = D\Delta u + f(u)$ . These equations model chemical reactions, population dynamics, and pattern formation in biology.

**Black-Scholes equation.** In mathematical finance, the Black-Scholes equation for option pricing is a parabolic PDE that can be transformed into the heat equation. It describes the



evolution of the price of a financial derivative.

**Image processing.** The heat equation is used in image processing for smoothing and denoising. Applying the heat equation to an image blurs it in a controlled way, removing high-frequency noise.



**Figure 6: Applications of the Heat Equation.** This figure illustrates various physical and mathematical applications of parabolic equations. **(Top left)** Heat diffusion in a rod with insulated ends, showing the decay of an initial temperature distribution. **(Top middle)** A traveling wave solution to the Fisher-KPP reaction-diffusion equation, modelling population spread. **(Top right)** The Black-Scholes equation for option pricing, showing how the value of a call option evolves as a function of stock price and time. **(Bottom left)** 2D diffusion from a point source, with concentration contours. **(Bottom middle)** Cooling of a heated plate with a step initial temperature distribution. **(Bottom right)** Comparison of solutions with different boundary conditions (Dirichlet, Neumann, mixed).

## 9 Conclusion

This chapter has introduced the fundamental concepts of parabolic partial differential equations, using the heat equation as the primary example. We have seen how the equation arises from physical principles, and we have explored the well-posedness of the initial/boundary value problem. The fundamental solution, or heat kernel, provides an explicit solution formula for the Cauchy problem and reveals the infinite speed of propagation and the smoothing property of the heat equation. The maximum principle, a powerful tool for parabolic equations, guarantees the uniqueness of solutions. The method of separation of variables and eigenfunction expansion provides explicit solutions for bounded domains and reveals the spectral structure of the heat operator. Energy methods and semigroup theory offer an abstract framework for understanding the long-time behaviour of solutions. The theory developed here for the heat equation lays the groundwork for the study of more general parabolic equations and provides a crucial point of comparison for the hyperbolic equations to be studied in the next chapter.

## References

- [1] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.
- [2] Lieberman, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific.
- [3] Friedman, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall.
- [4] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer.
- [5] Ladyženskaja, O. A., Solonnikov, V. A., & Ural'ceva, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society.

## Chapter 4

# Hyperbolic Equations: The Wave Equation and Propagation Phenomena

## 1 Introduction to Hyperbolic Equations and the Wave Equation

We now turn our attention to the third and final main class of second-order linear partial differential equations: **hyperbolic equations**. These equations model phenomena involving wave propagation, such as the vibrations of a string, the propagation of sound waves, or the dynamics of electromagnetic fields. The prototypical example of a hyperbolic equation is the **wave equation**.

Hyperbolic equations are fundamentally different from both elliptic and parabolic equations. They are second-order in both space and time, which gives rise to their characteristic wave-like behaviour. Unlike parabolic equations, which exhibit infinite speed of propagation and a strong smoothing effect, hyperbolic equations have a **finite speed of propagation**. This means that disturbances travel at a finite speed, and a change in the initial data at a point only affects a specific region of space-time, known as the **domain of influence**. Furthermore, hyperbolic equations do not smooth out singularities; instead, they **propagate singularities** along characteristic curves. This is why we can hear sharp sounds and see sharp images, as the waves that carry them do not get instantly blurred out.

A general second-order linear hyperbolic operator  $L$  is given by:

$$Lu = u_{tt} - \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower-order terms} \quad (4.1)$$

The operator is hyperbolic if the spatial part is elliptic, i.e., the matrix  $A(x,t) = (a_{ij}(x,t))$  is positive definite. The simplest hyperbolic equation is the wave equation, where  $A$  is the identity matrix and the lower-order terms are zero:

$$u_{tt} - c^2 \Delta u = 0 \quad (4.2)$$

where  $c$  is the wave speed.

This chapter will develop the classical theory for hyperbolic equations, focusing on the wave equation. We will derive the equation for a vibrating string, and then we will find the general solution in one dimension using d'Alembert's formula. This formula will make the concepts of domain of influence and finite speed of propagation precise. We will then discuss the conservation of energy for the wave equation. Finally, we will explore the solution in higher dimensions using Kirchhoff's formula and discuss Huygens' principle.

### 1.1 Derivation of the 1D Wave Equation

The one-dimensional wave equation can be derived by considering the transverse vibrations of a flexible, elastic string. We make the following simplifying assumptions:

1. The string is perfectly flexible and offers no resistance to bending.
2. The tension in the string is uniform and much larger than the force of gravity, so we can neglect gravity.
3. The vibrations are small and transverse, meaning each point on the string only moves vertically.

Let  $u(x, t)$  be the vertical displacement of the string at position  $x$  and time  $t$ . Consider a small segment of the string between  $x$  and  $x + \Delta x$ . The forces acting on this segment are the tension forces at its ends. Let  $T$  be the magnitude of the tension. The tension vectors at the ends are tangent to the string.

The vertical component of the tension force at  $x + \Delta x$  is  $T \sin \theta(x + \Delta x)$ , and at  $x$  it is  $-T \sin \theta(x)$ , where  $\theta$  is the angle the string makes with the horizontal. For small vibrations,  $\sin \theta \approx \tan \theta = u_x$ . So the net vertical force is:

$$F_v \approx T u_x(x + \Delta x, t) - T u_x(x, t) \quad (4.3)$$

By Newton's second law, this force must be equal to the mass of the segment times its acceleration. Let  $\rho$  be the linear density (mass per unit length) of the string. The mass of the segment is  $\rho \Delta x$ . The acceleration is  $u_{tt}$ . So we have:

$$\rho \Delta x u_{tt}(x, t) \approx T u_x(x + \Delta x, t) - T u_x(x, t) \quad (4.4)$$

Dividing by  $\Delta x$  and taking the limit as  $\Delta x \rightarrow 0$ , we get:

$$\rho u_{tt} = T u_{xx} \quad (4.5)$$

Rearranging this gives the one-dimensional wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \quad \text{where } c = \sqrt{\frac{T}{\rho}} \quad (4.6)$$

This derivation shows that the wave speed  $c$  is determined by the physical properties of the string: the tension and the density.

## 2 d'Alembert's Formula

For the one-dimensional wave equation, we can find a surprisingly simple and elegant general solution. This is known as d'Alembert's formula. Consider the wave equation:

$$u_{tt} - c^2 u_{xx} = 0 \quad (4.7)$$

We can factor the wave operator as:

$$(\partial_t - c \partial_x)(\partial_t + c \partial_x)u = 0 \quad (4.8)$$

Let  $v = u_t + c u_x$ . Then the equation becomes  $v_t - c v_x = 0$ . This is a first-order transport equation, and its general solution is  $v(x, t) = h(x + ct)$  for some function  $h$ . Now we have to solve:

$$u_t + c u_x = h(x + ct) \quad (4.9)$$

This is another first-order linear PDE. The general solution to the homogeneous part is  $f(x - ct)$ . A particular solution can be found by integrating  $h$ . The general solution to the wave equation is the sum of these, which can be written as:

$$u(x, t) = F(x - ct) + G(x + ct) \quad (4.10)$$

for arbitrary functions  $F$  and  $G$ . This is the **general solution of the 1D wave equation**. It shows that any solution can be written as the sum of a right-traveling wave  $F(x - ct)$  and a left-traveling wave  $G(x + ct)$ , both moving at speed  $c$ .

Now, consider the initial value problem (Cauchy problem) for the wave equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } x \in \mathbb{R}, t > 0 \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R} \\ u_t(x, 0) = h(x) & \text{for } x \in \mathbb{R} \end{cases} \quad (4.11)$$

We need to find the specific functions  $F$  and  $G$  that satisfy these initial conditions. Plugging in  $t = 0$  into the general solution and its time derivative, we get:

$$u(x, 0) = F(x) + G(x) = g(x) \quad (4.12)$$

$$u_t(x, 0) = -cF'(x) + cG'(x) = h(x) \quad (4.13)$$

Integrating the second equation gives  $-F(x) + G(x) = \frac{1}{c} \int_0^x h(s) ds + C$ . Solving this system for  $F$  and  $G$ , we find:

$$F(x) = \frac{1}{2}g(x) - \frac{1}{2c} \int_0^x h(s) ds - \frac{C}{2} \quad (4.14)$$

$$G(x) = \frac{1}{2}g(x) + \frac{1}{2c} \int_0^x h(s) ds + \frac{C}{2} \quad (4.15)$$

Substituting these back into the general solution  $u(x, t) = F(x - ct) + G(x + ct)$ , we obtain **d'Alembert's formula**:

$$u(x, t) = \frac{1}{2}[g(x - ct) + g(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s) ds \quad (4.16)$$

This remarkable formula gives the solution at any point  $(x, t)$  in terms of the initial position  $g$  and initial velocity  $h$ .

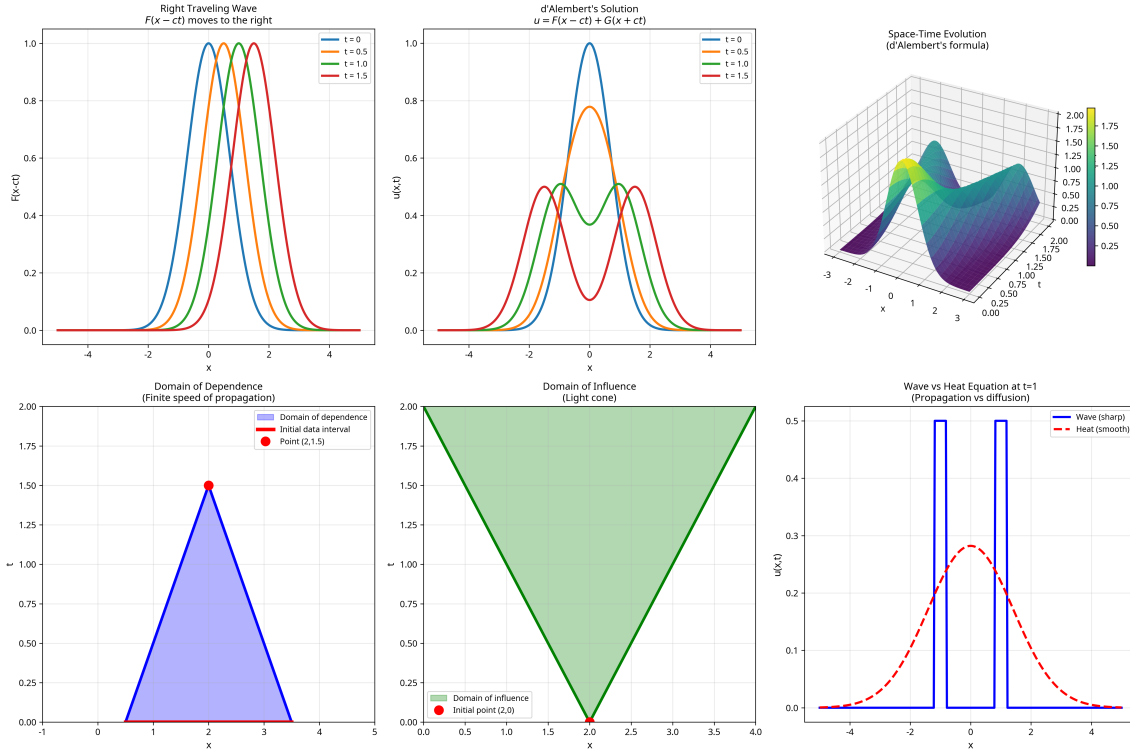


Figure 1: **d'Alembert's Formula and Traveling Waves.** This comprehensive figure illustrates the fundamental properties of the wave equation and d'Alembert's solution. **(Top left)** A right-traveling wave  $F(x - ct)$  moving to the right at speed  $c$ . **(Top middle)** Superposition of left and right traveling waves, demonstrating d'Alembert's general solution. **(Top right)** Space-time evolution of a wave showing the characteristic propagation pattern. **(Bottom left)** Domain of dependence: the solution at a point  $(x, t)$  depends only on the initial data in the interval  $[x - ct, x + ct]$ . **(Bottom middle)** Domain of influence: the forward light cone showing the region affected by initial data at a point. **(Bottom right)** Comparison with the heat equation, showing sharp propagation versus diffusive smoothing.

### 3 Domain of Influence and Finite Speed of Propagation

d'Alembert's formula clearly shows the concept of finite speed of propagation. The value of the solution  $u(x, t)$  at a point  $(x, t)$  depends only on the initial data on the interval  $[x - ct, x + ct]$ . This interval is called the **domain of dependence** of the point  $(x, t)$ .

Conversely, a point  $x_0$  on the initial line  $t = 0$  can only influence the solution in the region  $|x - x_0| \leq ct$ . This region is called the **domain of influence** of the point  $x_0$ . It is a cone in space-time, with its vertex at  $(x_0, 0)$  and its sides having slopes  $\pm 1/c$ . This is known as the **light cone**.

This is in stark contrast to the heat equation, where the solution at any point  $(x, t)$  with  $t > 0$  depends on the initial data everywhere. The wave equation has a finite speed of propagation, while the heat equation has an infinite speed of propagation.

### 4 Conservation of Energy

For the wave equation, there is a conserved quantity that can be interpreted as the energy of the wave. The energy is defined as the sum of the kinetic energy (from the velocity of the string) and the potential energy (from the stretching of the string).

The kinetic energy is given by:



$$K.E. = \frac{1}{2} \int \rho u_t^2 dx \quad (4.17)$$

The potential energy is the work done in stretching the string. The length of a small segment of the string is  $ds = \sqrt{1 + u_x^2} dx \approx (1 + \frac{1}{2}u_x^2) dx$ . The work done against the tension  $T$  is  $T(ds - dx) = \frac{1}{2}Tu_x^2 dx$ . So the total potential energy is:

$$P.E. = \frac{1}{2} \int Tu_x^2 dx \quad (4.18)$$

The total energy is:

$$E(t) = \frac{1}{2} \int (\rho u_t^2 + Tu_x^2) dx \quad (4.19)$$

Let's see if this energy is conserved. Differentiating with respect to time:

$$\frac{dE}{dt} = \int (\rho u_t u_{tt} + Tu_x u_{xt}) dx \quad (4.20)$$

Using the wave equation  $u_{tt} = c^2 u_{xx} = (T/\rho)u_{xx}$ , we have  $\rho u_{tt} = Tu_{xx}$ . Substituting this into the integral:

$$\frac{dE}{dt} = \int (Tu_{xx}u_t + Tu_x u_{xt}) dx = T \int (u_{xx}u_t + u_x u_{xt}) dx = T \int \frac{\partial}{\partial x}(u_x u_t) dx \quad (4.21)$$

By the Fundamental Theorem of Calculus, this integral is equal to  $[Tu_x u_t]$  evaluated at the boundaries. If we are on the whole real line and the solution has compact support, or if we have fixed (Dirichlet) or free (Neumann) boundary conditions, this boundary term is zero. Therefore:

$$\frac{dE}{dt} = 0 \quad (4.22)$$

This shows that the energy of the wave is conserved over time. This is a fundamental property of hyperbolic systems and is related to the time-reversibility of the wave equation. If we reverse time ( $t \rightarrow -t$ ), the wave equation is unchanged, and the energy is still conserved.

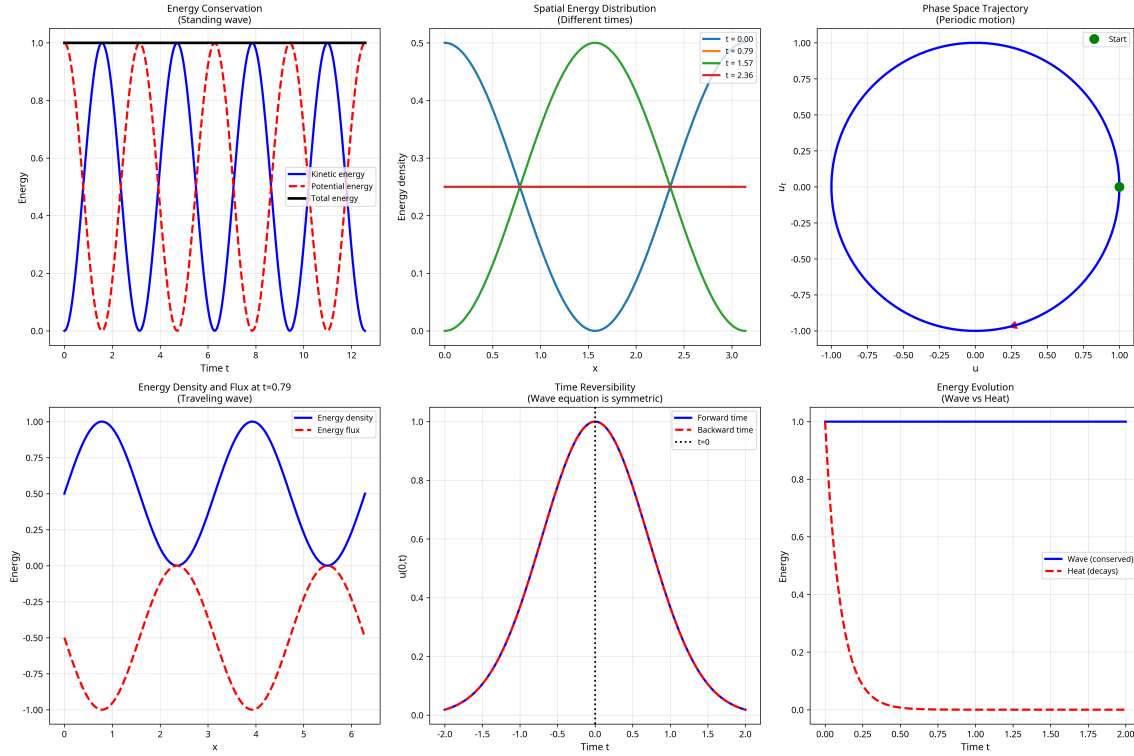


Figure 2: **Conservation of Energy for the Wave Equation.** This figure demonstrates the energy conservation property of the wave equation. **(Top left)** Periodic exchange between kinetic and potential energy for a standing wave, with total energy remaining constant. **(Top middle)** Spatial distribution of energy density at different times. **(Top right)** Phase space trajectory showing periodic motion and conservation. **(Bottom left)** Energy density and energy flux for a traveling wave. **(Bottom middle)** Time reversibility: the wave equation is symmetric under time reversal. **(Bottom right)** Comparison with the heat equation: wave energy is conserved while heat energy decays.

## 5 Separation of Variables for the Wave Equation

Similar to the heat equation, we can use the method of separation of variables to solve the wave equation on a bounded domain with homogeneous boundary conditions. This method leads to standing wave solutions.

Consider the wave equation on a rectangular domain  $\Omega = (0, a) \times (0, b)$  with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0 & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, y, 0) = g(x, y) & \text{in } \Omega \\ u_t(x, y, 0) = h(x, y) & \text{in } \Omega \end{cases} \quad (4.23)$$

We look for solutions of the form  $u(x, y, t) = X(x)Y(y)T(t)$ . Substituting into the wave equation and separating variables, we get:

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda \quad (4.24)$$

This leads to three separate eigenvalue problems. The spatial part gives:

$$-X'' = \mu X, \quad -Y'' = \nu Y, \quad \mu + \nu = \lambda \quad (4.25)$$

The solutions with the given boundary conditions are:

$$X_{mn}(x) = \sin\left(\frac{m\pi x}{a}\right) \quad (4.26)$$

$$Y_{mn}(y) = \sin\left(\frac{n\pi y}{b}\right) \quad (4.27)$$

with eigenvalues  $\mu_m = (m\pi/a)^2$  and  $\nu_n = (n\pi/b)^2$ . The spatial eigenfunctions are  $\phi_{mn}(x, y) = \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$ , and the eigenvalues are  $\lambda_{mn} = \pi^2(m^2/a^2 + n^2/b^2)$ .

The temporal part is  $T'' + c^2\lambda_{mn}T = 0$ , which has solutions:

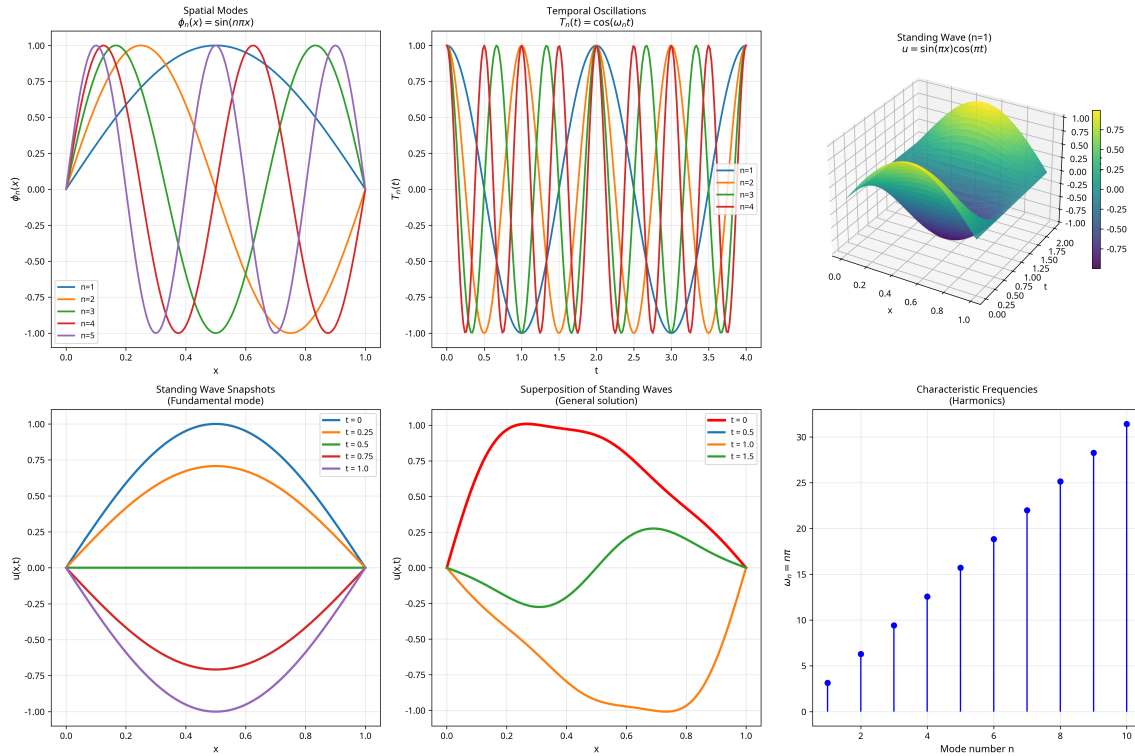
$$T_{mn}(t) = A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t) \quad (4.28)$$

where  $\omega_{mn} = c\sqrt{\lambda_{mn}}$  are the characteristic frequencies.

The general solution is a superposition of these standing waves:

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t)] \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (4.29)$$

The coefficients  $A_{mn}$  and  $B_{mn}$  are determined by the initial conditions.



**Figure 3: Standing Waves and Separation of Variables.** This figure illustrates the method of separation of variables for the wave equation. **(Top left)** The first five spatial modes (eigenfunctions) of the wave operator. **(Top middle)** Temporal oscillations of different modes at their characteristic frequencies. **(Top right)** Space-time evolution of a standing wave showing the fundamental mode. **(Bottom left)** Snapshots of a standing wave at different times, showing the oscillation pattern. **(Bottom middle)** Superposition of multiple standing waves to construct a general solution. **(Bottom right)** Characteristic frequencies (harmonics) showing the linear relationship with mode number.

## 6 The Wave Equation in Higher Dimensions

In three spatial dimensions, the wave equation is:

$$u_{tt} - c^2 \Delta u = 0, \quad \text{where } \Delta u = u_{xx} + u_{yy} + u_{zz} \quad (4.30)$$

The solution to the initial value problem in  $\mathbb{R}^3$  is given by **Kirchhoff's formula**. It is more complicated than d'Alembert's formula. If the initial data is  $u(x, 0) = g(x)$  and  $u_t(x, 0) = h(x)$ , the solution is:

$$u(x, t) = \frac{1}{4\pi c^2 t^2} \int_{\partial B(x, ct)} [th(y) + g(y) + \nabla g(y) \cdot (y - x)] dS(y) \quad (4.31)$$

This formula has a remarkable consequence known as **Huygens' principle**. The solution at a point  $(x, t)$  depends only on the initial data on the surface of the sphere of radius  $ct$  centred at  $x$ . It does not depend on the data inside the sphere. This means that a sharp pulse of initial data will result in a sharp pulse in the solution. There is no lingering effect, or “coda”. This is why we hear clean sounds in three dimensions.

In two spatial dimensions, the solution is given by a similar formula, but the integral is over the entire ball  $B(x, ct)$ , not just its boundary. This means that the solution at  $(x, t)$  depends on the initial data inside the sphere of influence. This leads to a lingering effect, and Huygens' principle in its sharp form does not hold in 2D. This is why a pebble dropped in a pond creates ripples that continue for some time.

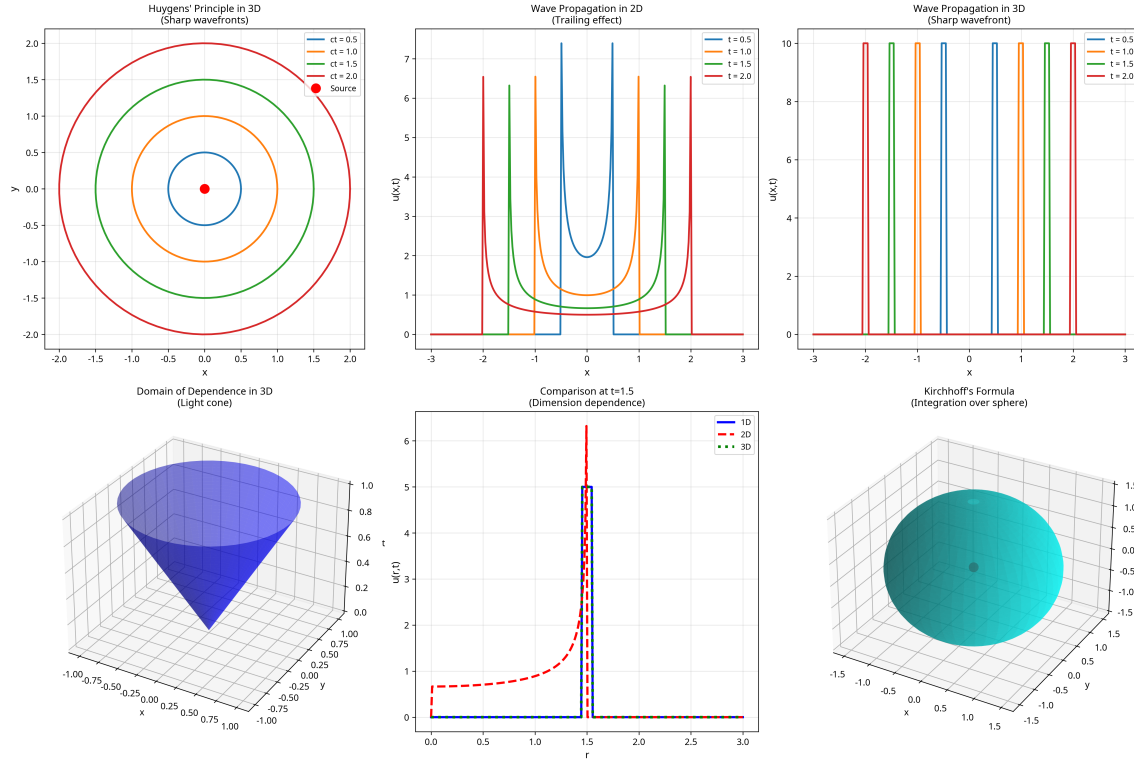


Figure 4: **Huygens' Principle and Wave Propagation in Higher Dimensions.** This figure illustrates Huygens' principle and the dimensional dependence of wave propagation. **(Top left)** Huygens' principle in 3D: sharp wavefronts propagate outward from a source. **(Top middle)** Wave propagation in 2D showing a trailing effect behind the wavefront. **(Top right)** Wave propagation in 3D showing a sharp wavefront without trailing. **(Bottom left)** The domain of dependence in 3D visualized as a light cone. **(Bottom middle)** Comparison of 1D, 2D, and 3D wave propagation at a fixed time. **(Bottom right)** Kirchhoff's formula: integration over the surface of a sphere.

## 7 Characteristics and the Method of Characteristics

The concept of **characteristics** is central to the theory of hyperbolic equations. For the wave equation, the characteristics are the curves along which information propagates. They are the solutions to the ordinary differential equations:

$$\frac{dx}{dt} = \pm c \quad (4.32)$$

These give the two families of characteristics:  $x = x_0 + ct$  (right-going) and  $x = x_0 - ct$  (left-going). The general solution  $u(x, t) = F(x - ct) + G(x + ct)$  shows that the solution is constant along each characteristic. The value of  $F$  is constant along the right-going characteristics, and the value of  $G$  is constant along the left-going characteristics.

The method of characteristics can be used to solve more general first-order PDEs and systems of first-order PDEs. For hyperbolic systems, the characteristics determine the domain of dependence and the domain of influence.

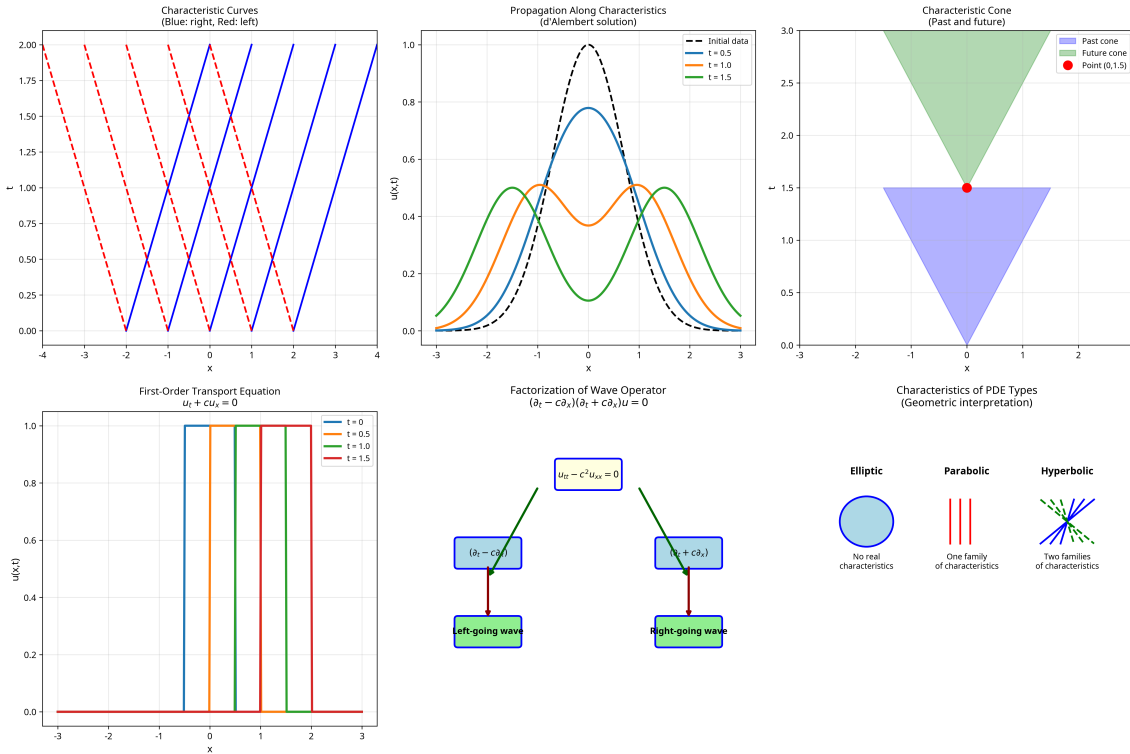


Figure 5: **Characteristics and the Method of Characteristics.** This figure illustrates the concept of characteristics for hyperbolic equations. **(Top left)** Characteristic curves for the wave equation: two families of lines with slopes  $\pm 1/c$ . **(Top middle)** Propagation of initial data along characteristics according to d'Alembert's solution. **(Top right)** The characteristic cone showing past and future light cones. **(Bottom left)** First-order transport equation showing propagation along a single family of characteristics. **(Bottom middle)** Factorization of the wave operator into two first-order operators. **(Bottom right)** Comparison of characteristics for elliptic, parabolic, and hyperbolic equations.

## 8 Applications of the Wave Equation

The wave equation and its generalizations arise in numerous physical contexts. We briefly survey some of the most important applications.

**Vibrating strings and membranes.** The most direct application is to the vibrations of strings (1D) and membranes (2D), such as musical instruments.

**Acoustic waves.** Sound waves in air, water, or other media are governed by the wave equation. The pressure or velocity perturbations satisfy the wave equation.

**Electromagnetic waves.** In vacuum, the electric and magnetic fields satisfy the wave equation. This is the basis of Maxwell's theory of electromagnetism and explains the propagation of light, radio waves, and other electromagnetic radiation.

**Seismic waves.** Earthquakes generate seismic waves that propagate through the Earth. There are two main types: P-waves (compression waves) and S-waves (shear waves), both governed by wave equations.

**Water waves.** Surface waves on water are approximately governed by wave equations, though the full theory involves nonlinear and dispersive effects.

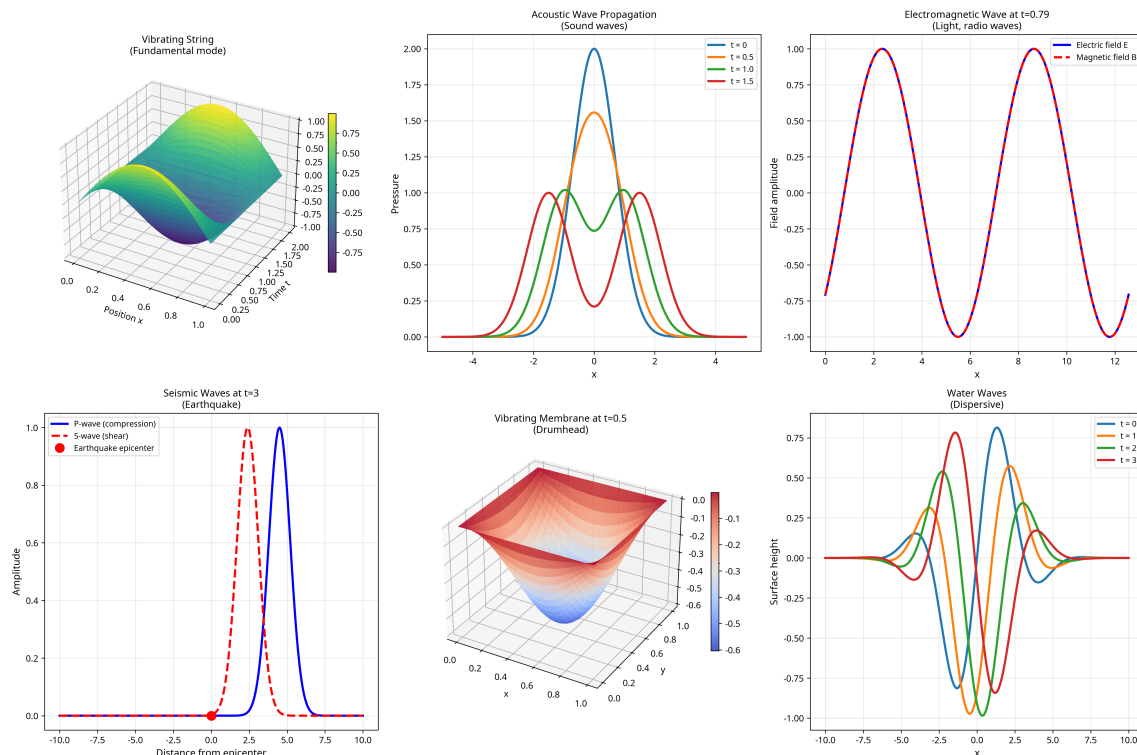


Figure 6: **Applications of the Wave Equation.** This figure illustrates various physical applications of hyperbolic equations. **(Top left)** Vibrating string showing the fundamental mode of oscillation. **(Top middle)** Acoustic wave propagation showing sound pulses traveling at the speed of sound. **(Top right)** Electromagnetic wave showing the electric and magnetic fields oscillating in phase. **(Bottom left)** Seismic waves: P-waves (compression) and S-waves (shear) propagating at different speeds. **(Bottom middle)** Vibrating membrane (drumhead) showing a 2D standing wave pattern. **(Bottom right)** Water waves showing dispersive propagation.

## 9 Conclusion

This chapter has provided an introduction to hyperbolic partial differential equations, with the wave equation as the central example. We have seen its derivation, its general solution in one dimension via d'Alembert's formula, and the crucial concepts of finite speed of propagation and domain of influence. The conservation of energy for the wave equation was demonstrated, highlighting its time-reversible nature. We also briefly explored the solution in higher dimensions and the significance of Huygens' principle. The method of separation of variables was used to find standing wave solutions on a bounded domain, and the concept of characteristics was introduced. The theory of hyperbolic equations is fundamental to our understanding of wave phenomena in physics and engineering, and it provides a stark contrast to the diffusive nature of parabolic equations and the steady-state character of elliptic equations.

## References

- [1] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.
- [2] John, F. (1982). *Partial Differential Equations* (4th ed.). Springer.
- [3] Strauss, W. A. (2007). *Partial Differential Equations: An Introduction* (2nd ed.). Wiley.



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- [4] Courant, R., & Hilbert, D. (1962). *Methods of Mathematical Physics, Volume II: Partial Differential Equations*. Wiley.
  - [5] Whitham, G. B. (1974). *Linear and Nonlinear Waves*. Wiley.

# Chapter 5

## Nonlinear Partial Differential Equations

### 1 Introduction to Nonlinear Partial Differential Equations

This chapter marks a significant transition from the linear theory developed in the previous chapters to the vast and complex world of **nonlinear partial differential equations**. While linear PDEs provide excellent models for many physical phenomena, a deeper understanding of the natural world requires grappling with nonlinearity. Nonlinearity arises when the principle of superposition fails, meaning the sum of two solutions is not necessarily a solution. This leads to a rich variety of new and challenging phenomena, such as the formation of shocks, solitons, turbulence, and pattern formation.

Nonlinear PDEs are notoriously difficult to solve. There is no general theory comparable to the one for linear equations. Instead, a diverse array of methods has been developed, each tailored to specific types of nonlinearities. This chapter will introduce some of the fundamental techniques used to study nonlinear PDEs, including methods from functional analysis, variational calculus, and bifurcation theory.

We will classify nonlinear PDEs into three main types:

1. **Semilinear equations:** The highest-order derivatives are linear, and the nonlinearity appears only in the lower-order terms. For example, a semilinear elliptic equation has the form:

$$-\Delta u = f(x, u, \nabla u) \quad (5.1)$$

2. **Quasilinear equations:** The coefficients of the highest-order derivatives depend on the solution and its lower-order derivatives. For example, a quasilinear elliptic equation has the form:

$$-\sum_{i,j=1}^n a_{ij}(x, u, \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \nabla u) \quad (5.2)$$

3. **Fully nonlinear equations:** The equation is nonlinear in its highest-order derivatives. For example, a fully nonlinear elliptic equation has the form:

$$F(x, u, \nabla u, D^2 u) = 0 \quad (5.3)$$

where  $F$  is nonlinear in the Hessian matrix  $D^2 u$ .

This chapter will focus primarily on semilinear and quasilinear equations, as they are more amenable to the methods we will develop. We will begin by exploring the use of fixed-point theorems to establish the existence of solutions. Then, we will introduce the powerful tools of variational methods and monotone operator theory. Finally, we will touch upon some important examples of nonlinear PDEs that arise in applications, such as the Navier-Stokes equations of fluid dynamics and reaction-diffusion equations from mathematical biology.

## 2 Fixed-Point Methods

One of the most fundamental approaches to solving nonlinear equations is to reformulate them as a fixed-point problem of the form  $u = T(u)$ , where  $T$  is a suitable operator on a Banach space. If we can show that  $T$  has a fixed point, then we have found a solution to our PDE.

### 2.1 The Contraction Mapping Principle

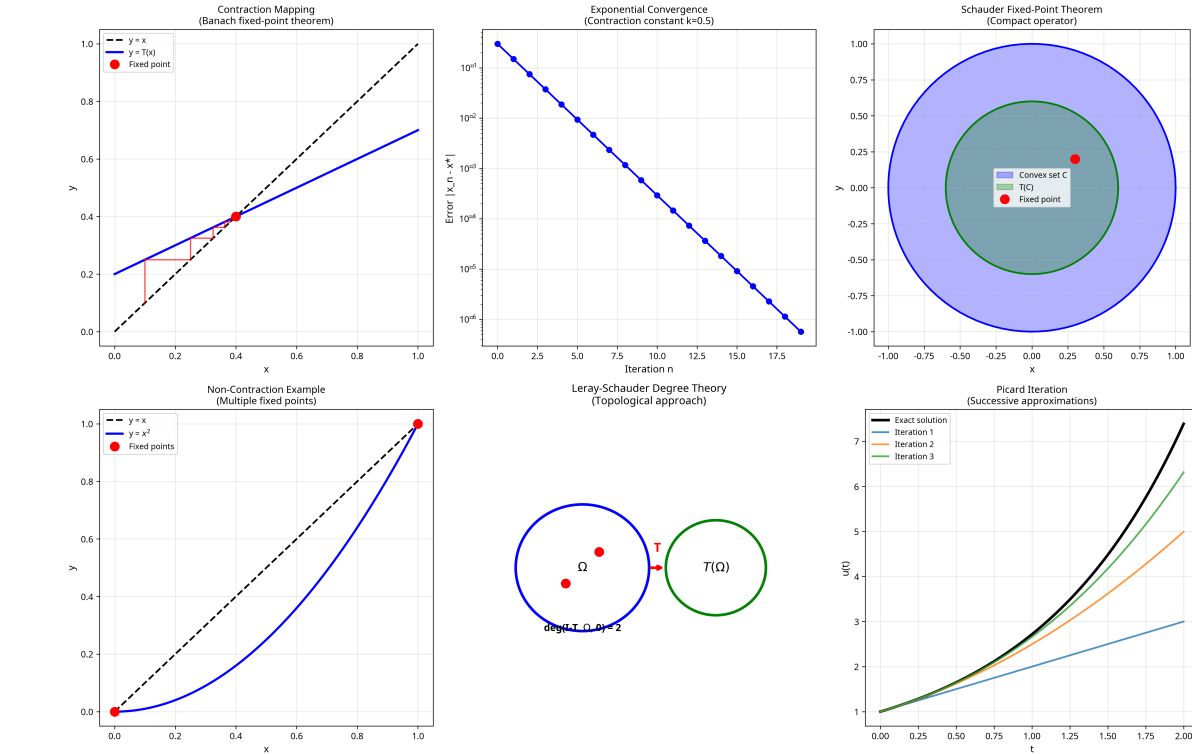
The simplest and most powerful fixed-point theorem is the **Contraction Mapping Principle**, also known as the Banach Fixed-Point Theorem. It states that if  $T$  is a contraction mapping on a complete metric space, then it has a unique fixed point.

**Theorem 2.1** (Contraction Mapping Principle). *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be a contraction mapping, i.e., there exists a constant  $0 \leq k < 1$  such that for all  $x, y \in X$ ,*

$$d(T(x), T(y)) \leq k d(x, y) \quad (5.4)$$

*Then  $T$  has a unique fixed point  $u^* \in X$ . Furthermore, for any  $u_0 \in X$ , the sequence defined by  $u_{n+1} = T(u_n)$  converges to  $u^*$ .*

This theorem is very useful for proving the existence and uniqueness of solutions to PDEs, especially for short times or for small data. The idea is to invert the linear part of the PDE to define an operator  $T$ , and then show that  $T$  is a contraction on a suitable function space.



**Figure 1: Fixed-Point Methods and Contraction Mappings.** This figure illustrates the fundamental concepts of fixed-point theory. **(Top left)** Contraction mapping showing convergence to a unique fixed point through successive iterations. **(Top middle)** Exponential convergence rate of the Picard iteration. **(Top right)** Schauder fixed-point theorem: a compact operator mapping a convex set to itself has a fixed point. **(Bottom left)** Non-contraction example with multiple fixed points. **(Bottom middle)** Leray-Schauder degree theory providing a topological approach to fixed-point problems. **(Bottom right)** Picard iteration for ODEs showing successive approximations converging to the exact solution.

## 2.2 Schauder Fixed-Point Theorem

For many nonlinear PDEs, the operator  $T$  is not a contraction, but it is compact. In this case, we can use a different class of fixed-point theorems, such as the **Schauder Fixed-Point Theorem**.

**Theorem 2.2** (Schauder Fixed-Point Theorem). *Let  $X$  be a Banach space and let  $C$  be a nonempty, closed, convex subset of  $X$ . If  $T : C \rightarrow C$  is a compact operator (i.e., it is continuous and maps bounded sets to precompact sets), then  $T$  has a fixed point.*

Unlike the Contraction Mapping Principle, the Schauder theorem does not guarantee uniqueness of the fixed point, nor does it provide a constructive method for finding it. However, it is a very powerful tool for proving the existence of solutions to a wide range of nonlinear PDEs, particularly elliptic equations.

## 2.3 Leray-Schauder Degree Theory

A more advanced tool is **Leray-Schauder degree theory**, which is a generalization of the Brouwer degree of a map to infinite-dimensional spaces. The Leray-Schauder degree is an integer associated with a compact operator and a domain, which counts the number of fixed points. If the degree is non-zero, then there must be at least one fixed point. This theory is particularly useful for proving the existence of solutions to systems of nonlinear equations and for studying bifurcation phenomena.

# 3 Variational Methods

Many nonlinear PDEs can be formulated as the Euler-Lagrange equation of a certain functional. In this case, solutions to the PDE correspond to critical points of the functional. This allows us to use the powerful tools of the **calculus of variations** to prove the existence of solutions.

## 3.1 The Euler-Lagrange Equation

Consider a functional of the form:

$$I(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx \quad (5.5)$$

where  $L$  is the Lagrangian. A function  $u$  is a critical point of  $I$  if the first variation of  $I$  at  $u$  is zero, i.e., for all smooth functions  $v$  with compact support in  $\Omega$ ,

$$\left. \frac{d}{d\epsilon} I(u + \epsilon v) \right|_{\epsilon=0} = 0 \quad (5.6)$$

Calculating the derivative and integrating by parts, we obtain the **Euler-Lagrange equation**:

$$\frac{\partial L}{\partial u} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial u_{x_i}} \right) = 0 \quad (5.7)$$

This is a second-order PDE for  $u$ . Thus, finding solutions to this PDE is equivalent to finding critical points of the functional  $I$ .

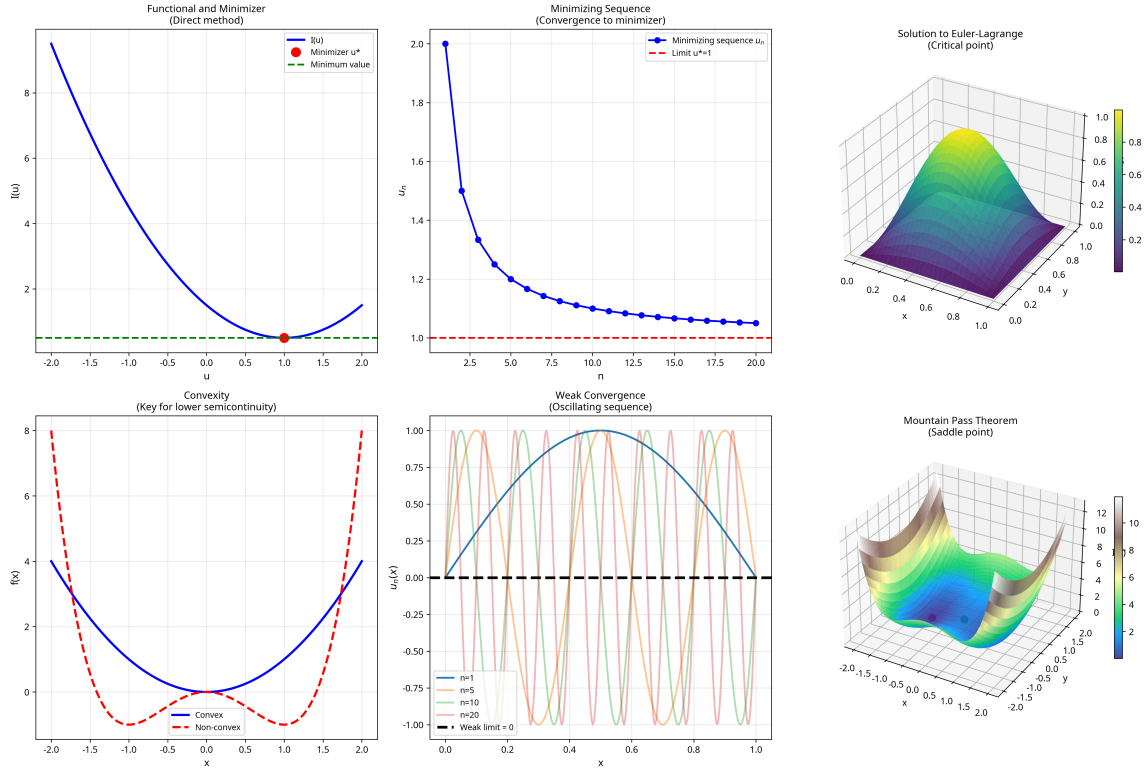


Figure 2: **Variational Methods and the Direct Method.** This figure illustrates the calculus of variations approach to nonlinear PDEs. **(Top left)** A functional with its minimizer, demonstrating the direct method. **(Top middle)** Minimizing sequence converging to the minimizer. **(Top right)** Solution to the Euler-Lagrange equation as a critical point. **(Bottom left)** Convexity property crucial for lower semicontinuity. **(Bottom middle)** Weak convergence of an oscillating sequence. **(Bottom right)** Mountain pass theorem landscape showing saddle points between two minima.

### 3.2 The Direct Method of the Calculus of Variations

The **direct method of the calculus of variations** is a powerful technique for proving the existence of a minimizer for a functional. The idea is to show that the functional is bounded below and then to find a minimizing sequence. The main difficulty is to show that the minimizing sequence converges to a limit that is indeed a minimizer.

The key steps are:

1. Show that the functional  $I$  is bounded below on a suitable function space  $X$ .
2. Let  $m = \inf_{u \in X} I(u)$  and let  $(u_n)$  be a minimizing sequence, i.e.,  $I(u_n) \rightarrow m$ .
3. Show that the minimizing sequence is bounded in  $X$ . By the Banach-Alaoglu theorem, we can extract a weakly convergent subsequence  $u_{n_k} \rightharpoonup u$  in  $X$ .
4. Show that the functional  $I$  is weakly lower semicontinuous, i.e.,  $\liminf_{k \rightarrow \infty} I(u_{n_k}) \geq I(u)$ . This is often the most difficult step and typically requires the Lagrangian  $L$  to be convex in the gradient variable.
5. Conclude that  $I(u) \leq m$ , and since  $m$  is the infimum, we must have  $I(u) = m$ . Thus,  $u$  is a minimizer.

This method is very effective for proving the existence of solutions to nonlinear elliptic equations.

## 4 Monotone Operator Theory

Another powerful framework for studying nonlinear PDEs is the theory of **monotone operators**. This theory was developed in the 1960s by Browder and Minty and provides a general framework for solving nonlinear equations of the form  $A(u) = f$ , where  $A$  is a nonlinear operator from a Banach space to its dual.

**Definition 4.1** (Monotone Operator). Let  $X$  be a Banach space and  $X^*$  its dual. An operator  $A : X \rightarrow X^*$  is said to be **monotone** if for all  $u, v \in X$ ,

$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad (5.8)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $X^*$  and  $X$ .

The main result of monotone operator theory is the **Browder-Minty theorem**, which gives conditions for the surjectivity of a monotone operator.

**Theorem 4.2** (Browder-Minty Theorem). *Let  $X$  be a reflexive Banach space and let  $A : X \rightarrow X^*$  be a monotone, coercive, and continuous operator. Then  $A$  is surjective, i.e., for every  $f \in X^*$ , there exists a solution  $u \in X$  to the equation  $A(u) = f$ .*

An operator  $A$  is coercive if  $\langle A(u), u \rangle / \|u\| \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ . This theorem is extremely powerful for proving the existence of weak solutions to a wide class of nonlinear elliptic and parabolic problems.

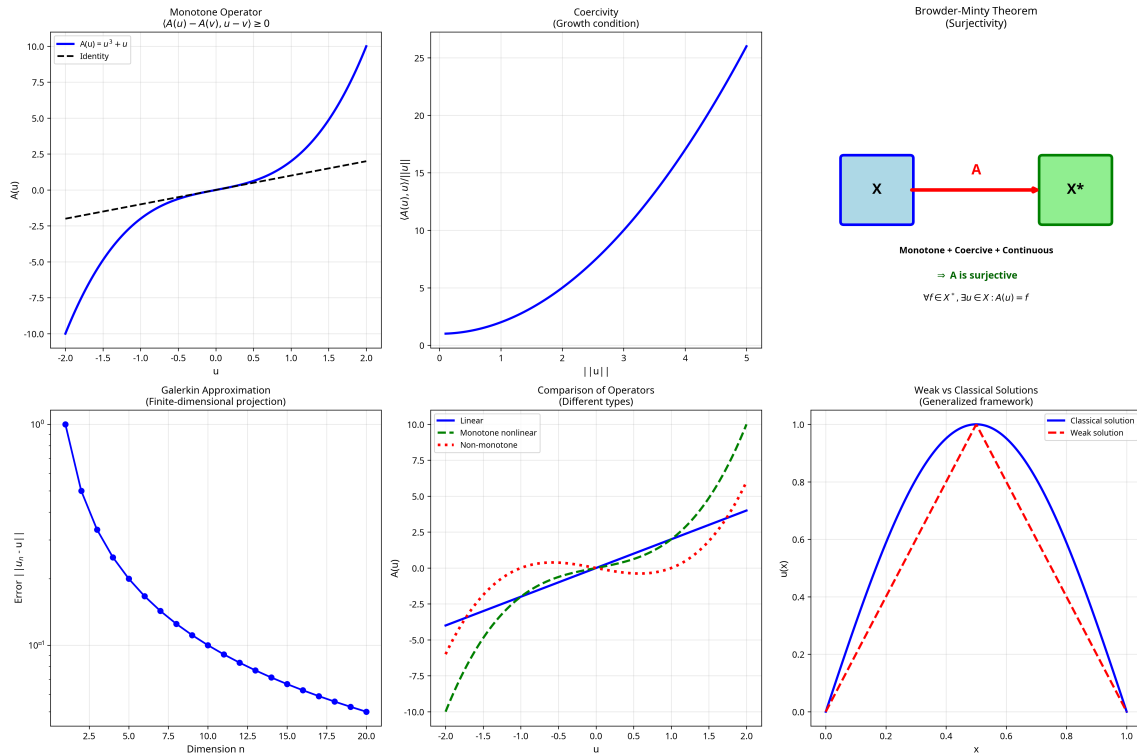


Figure 3: **Monotone Operator Theory**. This figure illustrates the theory of monotone operators. **(Top left)** Example of a monotone operator satisfying the monotonicity condition. **(Top middle)** Coercivity property showing growth at infinity. **(Top right)** Browder-Minty theorem: conditions for surjectivity of monotone operators. **(Bottom left)** Galerkin approximation showing convergence of finite-dimensional projections. **(Bottom middle)** Comparison of linear, monotone nonlinear, and non-monotone operators. **(Bottom right)** Weak versus classical solutions, illustrating the generalized framework.

## 5 Semilinear Elliptic Equations

Semilinear elliptic equations are among the most studied nonlinear PDEs. A typical example is:

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (5.9)$$

where  $f$  is a nonlinear function of  $u$ . Depending on the properties of  $f$ , various methods can be applied. If  $f$  has sublinear growth, the direct method of the calculus of variations can be used. If  $f$  is monotone, monotone operator theory applies. For more general nonlinearities, fixed-point methods or topological degree theory may be needed.

An important phenomenon in semilinear equations is **bifurcation**, where the structure of the solution set changes as a parameter varies. A simple example is the equation:

$$-\Delta u = \lambda u + u^3 \quad (5.10)$$

For small  $\lambda$ , the only solution is  $u = 0$ . But at a critical value of  $\lambda$ , non-trivial solutions branch off from the trivial solution. This is known as a pitchfork bifurcation.

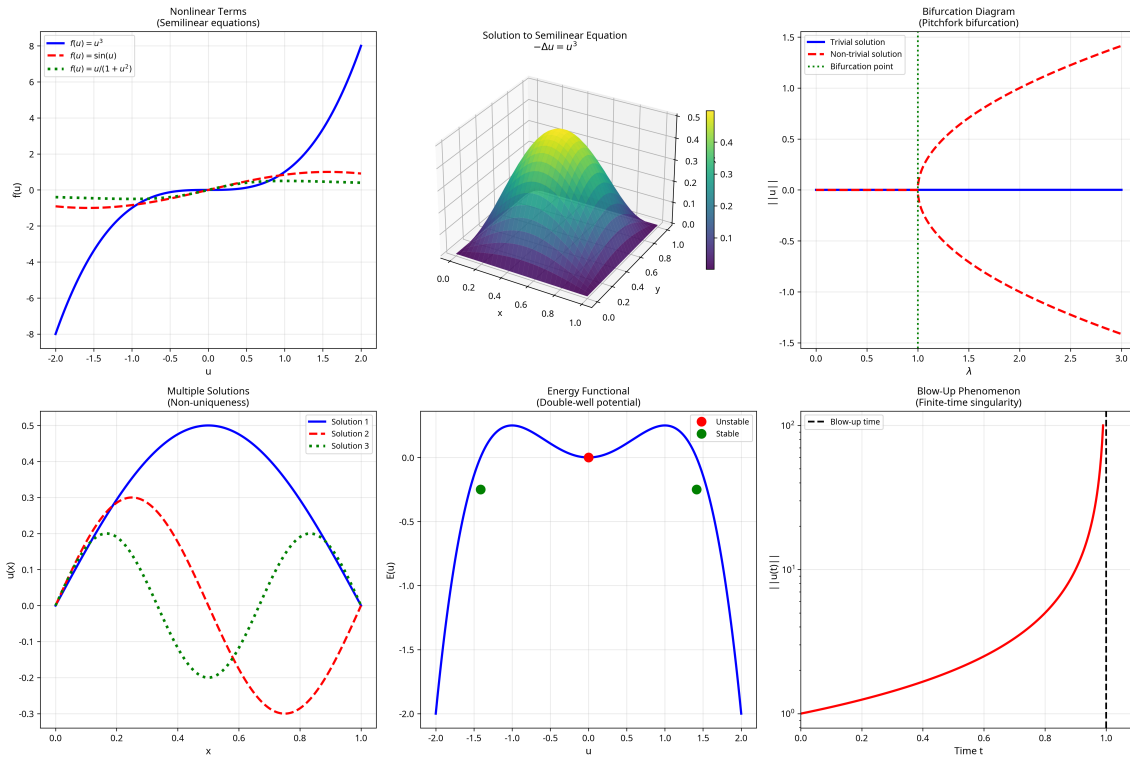


Figure 4: **Semilinear Elliptic Equations.** This figure illustrates various aspects of semilinear equations. **(Top left)** Different types of nonlinear terms. **(Top middle)** Solution to a semilinear equation. **(Top right)** Bifurcation diagram showing pitchfork bifurcation. **(Bottom left)** Multiple solutions demonstrating non-uniqueness. **(Bottom middle)** Energy functional with double-well potential. **(Bottom right)** Blow-up phenomenon showing finite-time singularity.

## 6 Reaction-Diffusion Equations

Reaction-diffusion equations model the interaction between diffusion and a local reaction process. A prototypical example is the **Fisher-KPP equation**:

$$u_t = Du_{xx} + f(u) \quad (5.11)$$



where  $f(u) = u(1 - u)$  is the logistic growth term. This equation models the spread of an advantageous gene in a population. It admits traveling wave solutions of the form  $u(x, t) = U(x - ct)$ , where  $c$  is the wave speed.

Reaction-diffusion systems can also exhibit pattern formation through the **Turing instability**. When two chemicals with different diffusion rates interact, the uniform steady state can become unstable, leading to spatially periodic patterns. This mechanism is thought to be responsible for many patterns in nature, such as animal coat patterns and vegetation patterns in arid ecosystems.

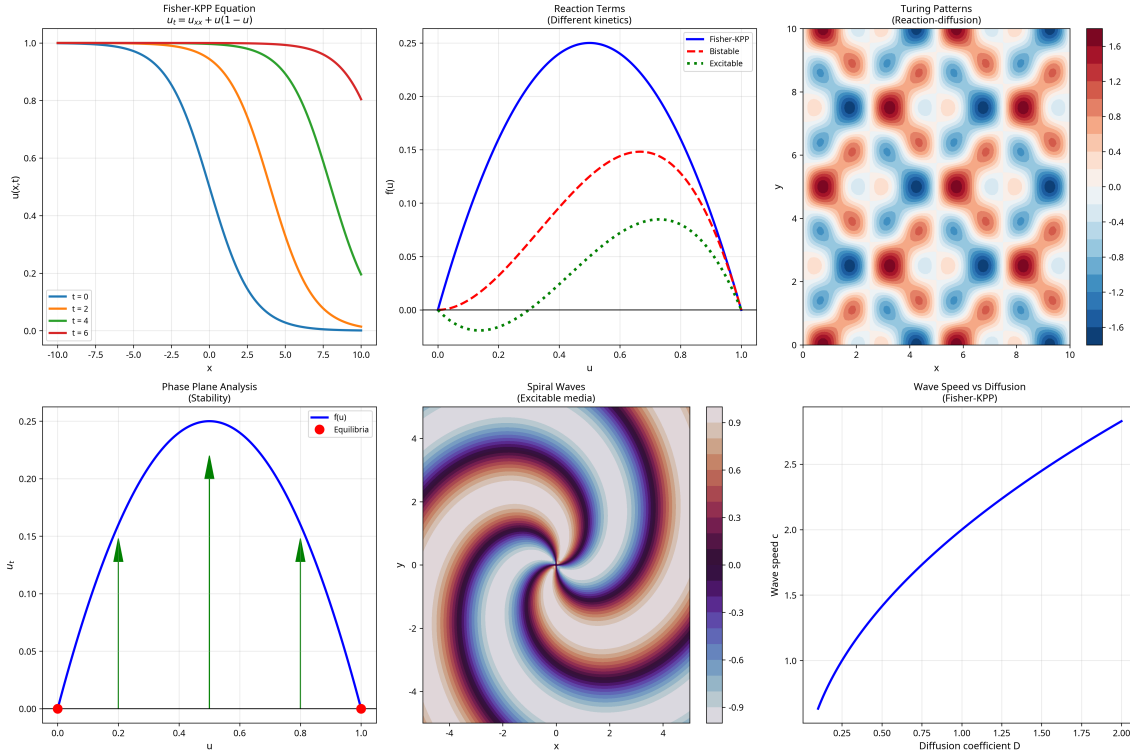


Figure 5: **Reaction-Diffusion Equations.** This figure illustrates reaction-diffusion phenomena. **(Top left)** Fisher-KPP equation showing traveling wave solutions. **(Top middle)** Different reaction terms (Fisher-KPP, bistable, excitable). **(Top right)** Turing patterns arising from diffusion-driven instability. **(Bottom left)** Phase plane analysis showing stability of equilibria. **(Bottom middle)** Spiral waves in excitable media. **(Bottom right)** Wave speed as a function of diffusion coefficient.

## 7 Navier-Stokes Equations and Other Nonlinear PDEs

The **Navier-Stokes equations** describe the motion of viscous incompressible fluids:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad (5.12)$$

where  $\mathbf{u}$  is the velocity,  $p$  is the pressure,  $\nu$  is the kinematic viscosity, and  $\mathbf{f}$  is an external force. The nonlinear term  $(\mathbf{u} \cdot \nabla) \mathbf{u}$  represents the convective acceleration and is responsible for the rich dynamics of fluid flow, including turbulence.

The global existence and smoothness of solutions to the 3D Navier-Stokes equations is one of the most important open problems in mathematics. While weak solutions are known to exist, it is not known whether they remain smooth for all time or whether singularities can form.

Other important nonlinear PDEs include the **minimal surface equation**, which arises from the problem of finding a surface of minimal area with a given boundary, and the **Korteweg-de Vries (KdV) equation**, which models shallow water waves and admits soliton solutions.

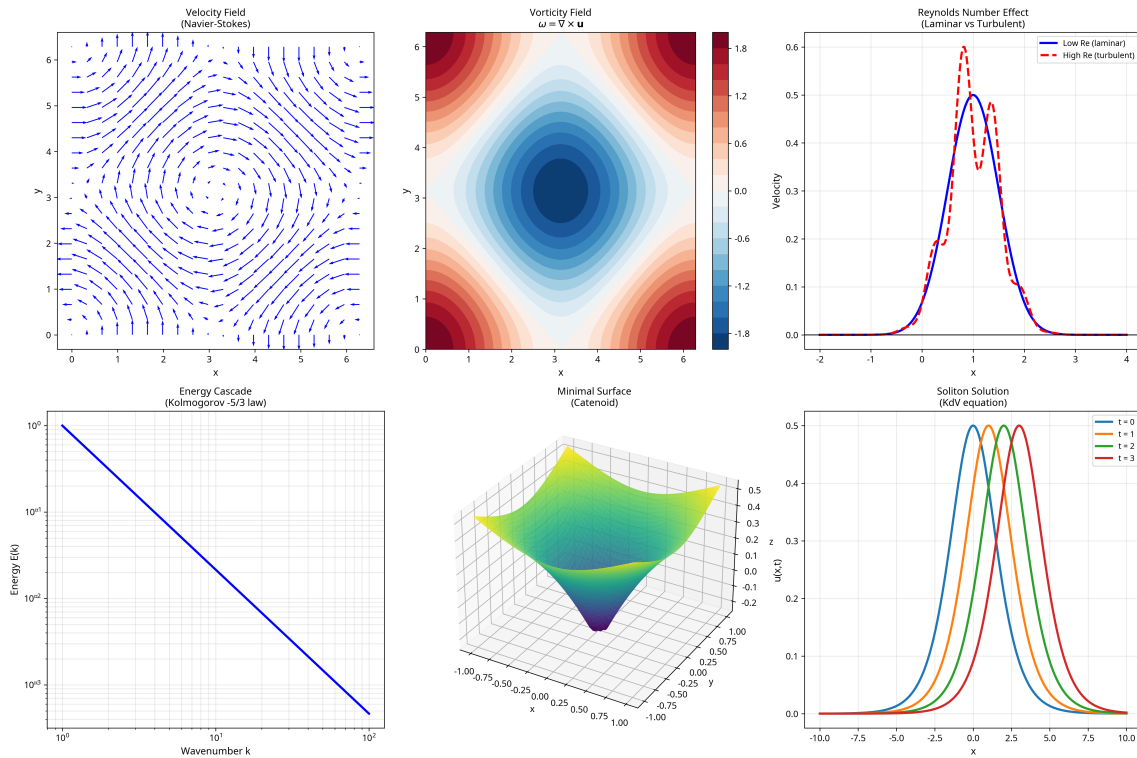


Figure 6: **Navier-Stokes and Other Nonlinear PDEs.** This figure illustrates various nonlinear phenomena. **(Top left)** Velocity field from Navier-Stokes equations. **(Top middle)** Vorticity field showing rotational structures. **(Top right)** Reynolds number effect: transition from laminar to turbulent flow. **(Bottom left)** Energy cascade in turbulence (Kolmogorov  $-5/3$  law). **(Bottom middle)** Minimal surface (catenoid). **(Bottom right)** Soliton solution to the KdV equation.

## 8 Conclusion

This chapter has provided a glimpse into the vast and fascinating world of nonlinear partial differential equations. We have introduced several powerful methods for proving the existence of solutions, including fixed-point theorems, variational methods, and monotone operator theory. We have also seen a variety of important examples of nonlinear PDEs that arise in applications. The study of nonlinear PDEs is a very active area of research, and many fundamental questions remain open. The methods and ideas introduced in this chapter provide a foundation for further exploration of this rich and challenging field.

## References

- [1] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.
- [2] Gilbarg, D., & Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. Springer.

- [3] Struwe, M. (2008). *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems* (4th ed.). Springer.
- [4] Zeidler, E. (1990). *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*. Springer.
- [5] Smoller, J. (1994). *Shock Waves and Reaction-Diffusion Equations* (2nd ed.). Springer.

# Chapter 6

## Sobolev Spaces and Weak Solutions

### 1 Introduction to Sobolev Spaces and Weak Solutions

The classical theory of partial differential equations, as discussed in the previous chapters, is primarily concerned with finding smooth, or *classical*, solutions. However, many PDEs that arise from physical models or variational problems do not possess classical solutions. The solutions may, for instance, have corners or other singularities. This limitation led to the development of the concept of **weak solutions**, which are solutions in a generalized sense. The natural setting for the study of weak solutions is the theory of **Sobolev spaces**, which are function spaces designed to handle functions with weak derivatives.

Introduced by the Soviet mathematician Sergei Sobolev in the 1930s, these spaces have become an indispensable tool in the modern theory of PDEs. They provide a natural framework for proving the existence, uniqueness, and regularity of weak solutions to a vast range of linear and nonlinear PDEs. This chapter provides a rigorous introduction to the theory of Sobolev spaces and their application to elliptic boundary value problems.

#### 1.1 Weak Derivatives

The key idea behind Sobolev spaces is to generalize the notion of a derivative. Instead of requiring a function to be differentiable in the classical sense, we define its derivative in a *weak* sense by using integration by parts.

**Definition 1.1** (Weak Derivative). Let  $u \in L^1_{loc}(\Omega)$ , where  $\Omega$  is an open set in  $\mathbb{R}^n$ . Let  $\alpha$  be a multi-index. We say that a function  $v \in L^1_{loc}(\Omega)$  is the  $\alpha$ -th **weak partial derivative** of  $u$ , and we write  $D^\alpha u = v$ , if for every test function  $\phi \in C_c^\infty(\Omega)$  (i.e., an infinitely differentiable function with compact support in  $\Omega$ ),

$$\int_{\Omega} u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v(x) \phi(x) dx \quad (6.1)$$

If a weak derivative exists, it is unique (up to a set of measure zero). If a function is continuously differentiable, its classical derivative is also its weak derivative. However, a function can have a weak derivative even if it is not differentiable everywhere in the classical sense.

**Example 1.2** (Weak derivative of  $|x|$ ). Consider the function  $u(x) = |x|$  on the interval  $\Omega = (-1, 1)$ . This function is not differentiable at  $x = 0$ . However, it has a weak derivative given by the sign function:

$$v(x) = \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (6.2)$$

To see this, we integrate by parts:

$$\begin{aligned}
 \int_{-1}^1 |x| \phi'(x) dx &= \int_{-1}^0 -x \phi'(x) dx + \int_0^1 x \phi'(x) dx \\
 &= [-x \phi(x)]_{-1}^0 + \int_{-1}^0 \phi(x) dx + [x \phi(x)]_0^1 - \int_0^1 \phi(x) dx \\
 &= \int_{-1}^0 \phi(x) dx - \int_0^1 \phi(x) dx \\
 &= - \int_{-1}^1 \operatorname{sgn}(x) \phi(x) dx
 \end{aligned}$$

since  $\phi(-1) = \phi(1) = 0$ . Thus, the weak derivative of  $|x|$  is  $\operatorname{sgn}(x)$ .

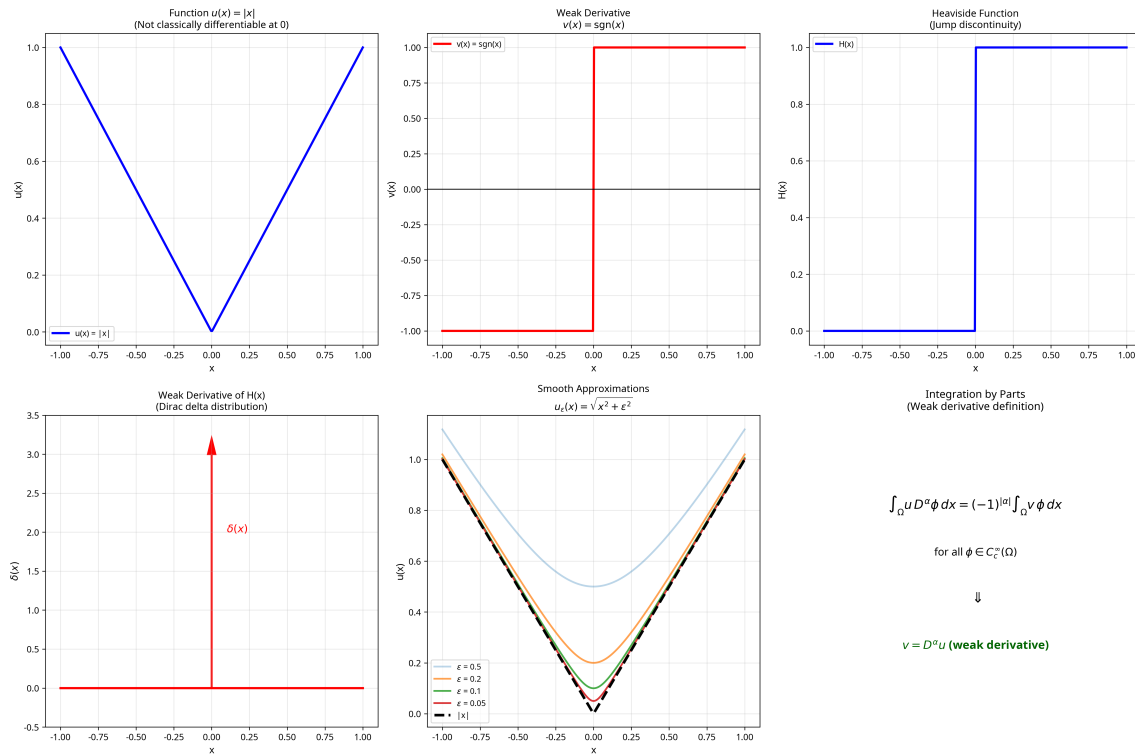


Figure 1: **Weak Derivatives and Sobolev Spaces.** This figure illustrates the concept of weak derivatives. **(Top left)** The function  $u(x) = |x|$  which is not classically differentiable at the origin. **(Top middle)** Its weak derivative  $v(x) = \operatorname{sgn}(x)$ . **(Top right)** The Heaviside function with a jump discontinuity. **(Bottom left)** The weak derivative of the Heaviside function is the Dirac delta distribution. **(Bottom middle)** Smooth approximations to  $|x|$  converging to the non-smooth function. **(Bottom right)** The definition of weak derivative via integration by parts.

## 2 Sobolev Spaces

With the concept of weak derivatives, we can now define Sobolev spaces.

**Definition 2.1** (Sobolev Space  $W^{k,p}(\Omega)$ ). Let  $k$  be a non-negative integer and  $1 \leq p \leq \infty$ . The **Sobolev space**  $W^{k,p}(\Omega)$  is the set of all functions  $u \in L^p(\Omega)$  such that for every multi-index  $\alpha$  with  $|\alpha| \leq k$ , the weak derivative  $D^\alpha u$  exists and belongs to  $L^p(\Omega)$ .

The Sobolev space  $W^{k,p}(\Omega)$  is a Banach space when equipped with the norm:

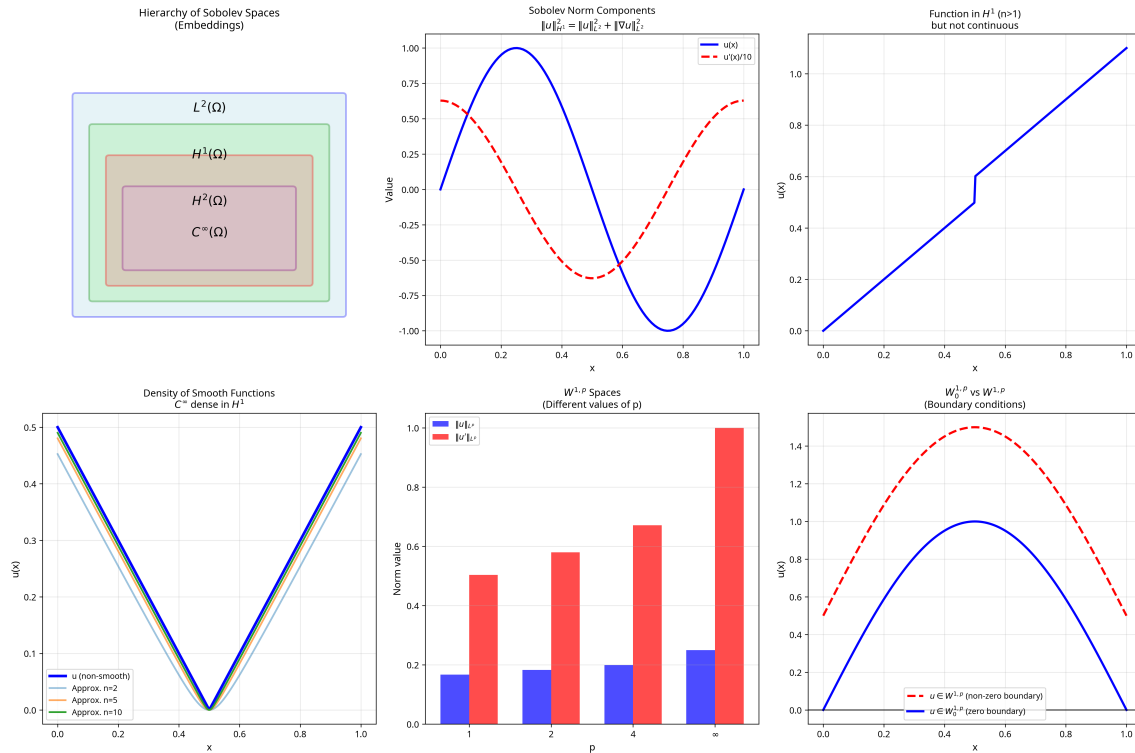
$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty \quad (6.3)$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} \quad \text{for } p = \infty \quad (6.4)$$

In the special case  $p = 2$ , the Sobolev spaces  $W^{k,2}(\Omega)$  are Hilbert spaces, and they are usually denoted by  $H^k(\Omega)$ . The inner product on  $H^k(\Omega)$  is given by:

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha u(x) D^\alpha v(x) dx \quad (6.5)$$



**Figure 2: Sobolev Spaces and Norms.** This figure illustrates the structure of Sobolev spaces. **(Top left)** Hierarchy of Sobolev spaces showing nested embeddings. **(Top middle)** Components of the Sobolev norm. **(Top right)** Example of a function in  $H^1$  (for  $n > 1$ ) that is not continuous. **(Bottom left)** Density of smooth functions in Sobolev spaces. **(Bottom middle)** Comparison of  $W^{1,p}$  norms for different values of  $p$ . **(Bottom right)** Distinction between  $W_0^{1,p}$  and  $W^{1,p}$  based on boundary conditions.

## 2.1 Properties of Sobolev Spaces

Sobolev spaces have many important properties that make them suitable for the study of PDEs.

- **Completeness:** Sobolev spaces are complete, i.e., they are Banach spaces (or Hilbert spaces for  $p = 2$ ). This is crucial for using functional analysis tools.

- **Reflexivity and Separability:** If  $1 < p < \infty$ , then  $W^{k,p}(\Omega)$  is a reflexive and separable Banach space.
- **Density:** If  $\Omega$  is sufficiently regular (e.g., has a Lipschitz boundary), then the space of smooth functions  $C^\infty(\overline{\Omega})$  is dense in  $W^{k,p}(\Omega)$ . This means that any Sobolev function can be approximated by a sequence of smooth functions.
- **Sobolev spaces with zero boundary values:** We define  $W_0^{k,p}(\Omega)$  as the closure of  $C_c^\infty(\Omega)$  in the  $W^{k,p}(\Omega)$  norm. Functions in  $W_0^{k,p}(\Omega)$  can be thought of as functions in  $W^{k,p}(\Omega)$  that are zero on the boundary  $\partial\Omega$  in a generalized sense.

### 3 Sobolev Embedding Theorems

One of the most important aspects of Sobolev space theory is the relationship between different Sobolev spaces and between Sobolev spaces and classical function spaces like  $C^k(\Omega)$  or  $L^q(\Omega)$ . These relationships are described by the **Sobolev embedding theorems**.

The Sobolev embedding theorems tell us that if a function has enough weak derivatives in  $L^p$ , then it must be more regular, for example, continuous or even continuously differentiable. The amount of extra regularity depends on the dimension  $n$  and the integrability parameter  $p$ .

**Theorem 3.1** (Gagliardo-Nirenberg-Sobolev Inequality). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with a  $C^1$  boundary. Let  $u \in W^{1,p}(\Omega)$  with  $1 \leq p < n$ . Then  $u \in L^{p^*}(\Omega)$ , where  $p^* = \frac{np}{n-p}$  is the Sobolev conjugate of  $p$ . Moreover, there exists a constant  $C$  depending only on  $n$  and  $p$  such that:*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad (6.6)$$

for all  $u \in W_0^{1,p}(\Omega)$ .

This theorem shows that we gain integrability for functions in  $W^{1,p}(\Omega)$ . By applying this theorem iteratively, we can obtain embeddings into spaces of continuous functions.

**Theorem 3.2** (Sobolev Embedding Theorem). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with a  $C^1$  boundary. Let  $u \in W^{k,p}(\Omega)$ .*

- If  $kp < n$ , then  $u \in L^q(\Omega)$  for all  $1 \leq q \leq \frac{np}{n-kp}$ .
- If  $kp = n$ , then  $u \in L^q(\Omega)$  for all  $1 \leq q < \infty$ .
- If  $kp > n$ , then  $u \in C^{k-[n/p]-1,\gamma}(\overline{\Omega})$  for some  $\gamma > 0$ , where  $[n/p]$  is the integer part of  $n/p$ . This means that  $u$  is Hölder continuous.

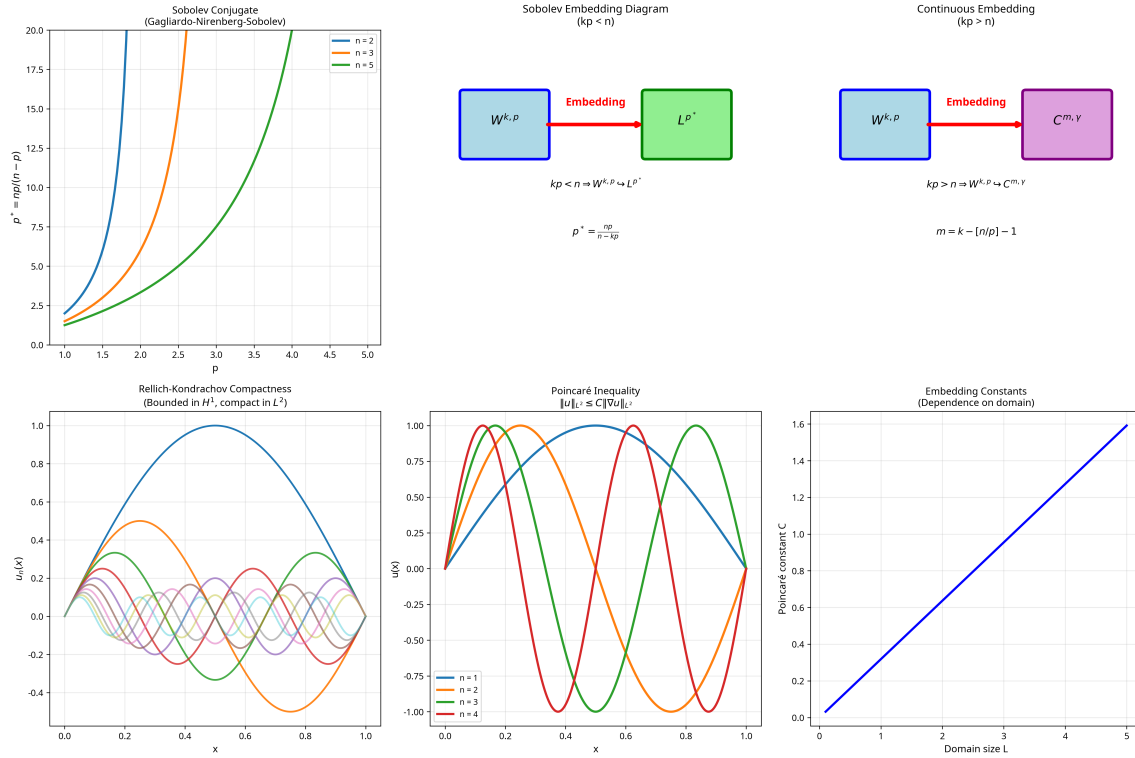


Figure 3: **Sobolev Embeddings.** This figure illustrates Sobolev embedding theorems. **(Top left)** The Sobolev conjugate  $p^*$  as a function of  $p$  for different dimensions  $n$ . **(Top middle)** Embedding diagram showing  $W^{k,p} \hookrightarrow L^{p^*}$  when  $kp < n$ . **(Top right)** Continuous embedding into Hölder spaces when  $kp > n$ . **(Bottom left)** Rellich-Kondrachov compactness: bounded sequences in  $H^1$  have convergent subsequences in  $L^2$ . **(Bottom middle)** Poincaré inequality relating  $L^2$  and gradient norms. **(Bottom right)** Embedding constants depend on domain size.

### 3.1 Compact Embeddings: Rellich-Kondrachov Theorem

For the direct method of the calculus of variations, we need not just a bounded sequence to have a weakly convergent subsequence, but we often need the convergence to be strong in some sense. This is provided by the **Rellich-Kondrachov theorem**, which gives conditions for the embedding of a Sobolev space into another to be compact.

**Theorem 3.3** (Rellich-Kondrachov Theorem). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with a  $C^1$  boundary. Let  $1 \leq p < n$ . Then the embedding of  $W^{1,p}(\Omega)$  into  $L^q(\Omega)$  is compact for all  $1 \leq q < p^* = \frac{np}{n-p}$ .*

This theorem is extremely useful. It implies that if a sequence is bounded in  $W^{1,p}(\Omega)$ , then we can extract a subsequence that converges strongly in  $L^q(\Omega)$  for  $q < p^*$ . This is often the key step in proving the existence of weak solutions to nonlinear PDEs.

## 4 Trace Theory

Another important question is how to define the value of a Sobolev function on the boundary of the domain. Since Sobolev functions are only defined up to a set of measure zero, the value on the boundary (which has measure zero) is not well-defined. **Trace theory** provides a way to give a precise meaning to the boundary values of Sobolev functions.

**Theorem 4.1** (Trace Theorem). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with a  $C^1$  boundary. There exists a bounded linear operator  $T : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ , called the **trace operator**, such that:*



- $T(u) = u|_{\partial\Omega}$  if  $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ .
- $\|T(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$  for some constant  $C$ .

The function  $T(u)$  is called the **trace** of  $u$  on  $\partial\Omega$ . The kernel of the trace operator is the space  $W_0^{1,p}(\Omega)$ .

This theorem allows us to make sense of boundary conditions for weak solutions. For example, a weak solution  $u$  to a PDE with Dirichlet boundary condition  $u = g$  on  $\partial\Omega$  is a function in  $W^{1,p}(\Omega)$  such that its trace  $T(u)$  is equal to  $g$ .

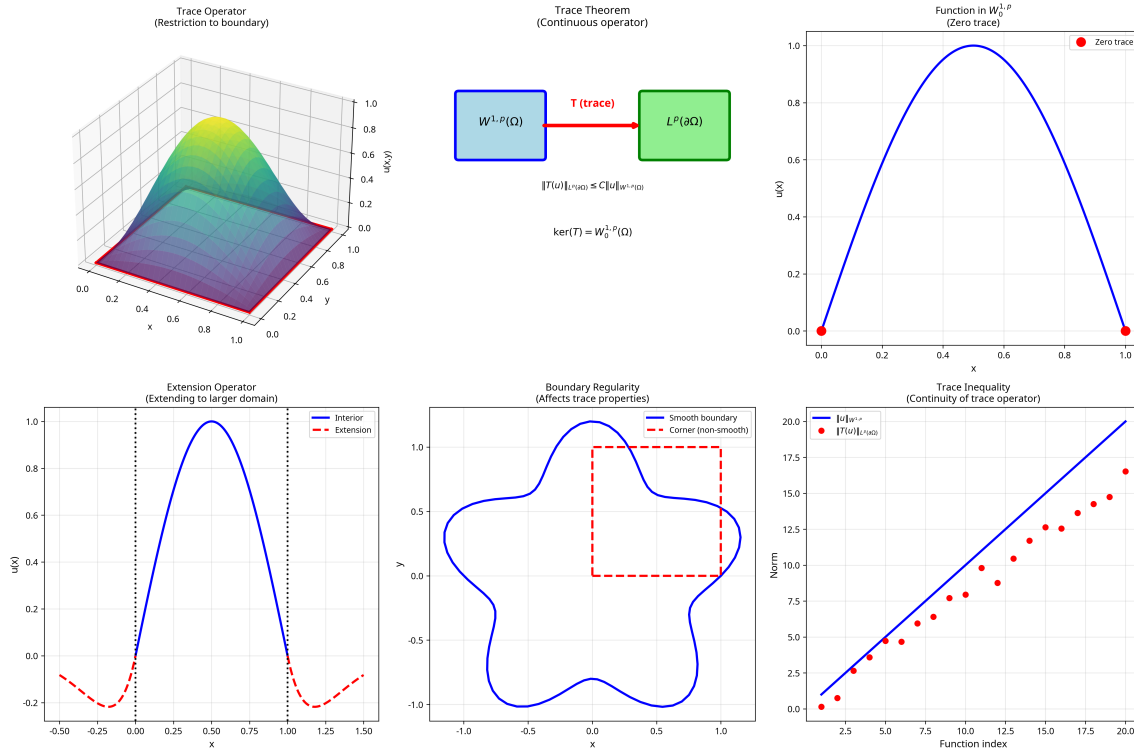


Figure 4: **Trace Theory.** This figure illustrates trace theory for Sobolev functions. **(Top left)** Trace operator concept: restriction of a function to the boundary. **(Top middle)** Trace theorem diagram showing the continuous operator from  $W^{1,p}$  to  $L^p(\partial\Omega)$ . **(Top right)** Function in  $W_0^{1,p}$  with zero trace. **(Bottom left)** Extension operator for extending functions from interior to exterior. **(Bottom middle)** Boundary regularity affects trace properties. **(Bottom right)** Trace inequality demonstrating continuity of the trace operator.

## 5 Application to Elliptic Equations: Weak Solutions

We now apply the machinery of Sobolev spaces to study the existence and uniqueness of weak solutions to elliptic boundary value problems. Consider the Poisson equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (6.7)$$

where  $f \in L^2(\Omega)$ .

To find a weak solution, we multiply the equation by a test function  $v \in C_c^\infty(\Omega)$  and integrate by parts:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad (6.8)$$

This leads to the following definition of a weak solution.

**Definition 5.1** (Weak Solution). A function  $u \in H_0^1(\Omega)$  is a **weak solution** of the Dirichlet problem for the Poisson equation if for all  $v \in H_0^1(\Omega)$ ,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad (6.9)$$

We can prove the existence and uniqueness of a weak solution using the **Lax-Milgram theorem**. Let  $H = H_0^1(\Omega)$ . We define a bilinear form  $B : H \times H \rightarrow \mathbb{R}$  and a linear functional  $F : H \rightarrow \mathbb{R}$  by:

$$\begin{aligned} B(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ F(v) &= \int_{\Omega} f v \, dx \end{aligned}$$

The weak formulation is then equivalent to finding  $u \in H$  such that  $B(u, v) = F(v)$  for all  $v \in H$ .

The Lax-Milgram theorem states that if  $B$  is bounded and coercive, then there exists a unique solution. We can easily check that these conditions are satisfied.

- **Boundedness:** By the Cauchy-Schwarz inequality,

$$|B(u, v)| = \left| \int_{\Omega} \nabla u \cdot \nabla v \, dx \right| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \|v\|_{H^1}$$

- **Coercivity:** By the Poincaré inequality, there exists a constant  $C$  such that  $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$  for all  $u \in H_0^1(\Omega)$ . Then,

$$B(u, u) = \|\nabla u\|_{L^2}^2 \geq \frac{1}{1 + C^2} (\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) = \frac{1}{1 + C^2} \|u\|_{H^1}^2$$

Thus, the Lax-Milgram theorem guarantees the existence of a unique weak solution  $u \in H_0^1(\Omega)$ .

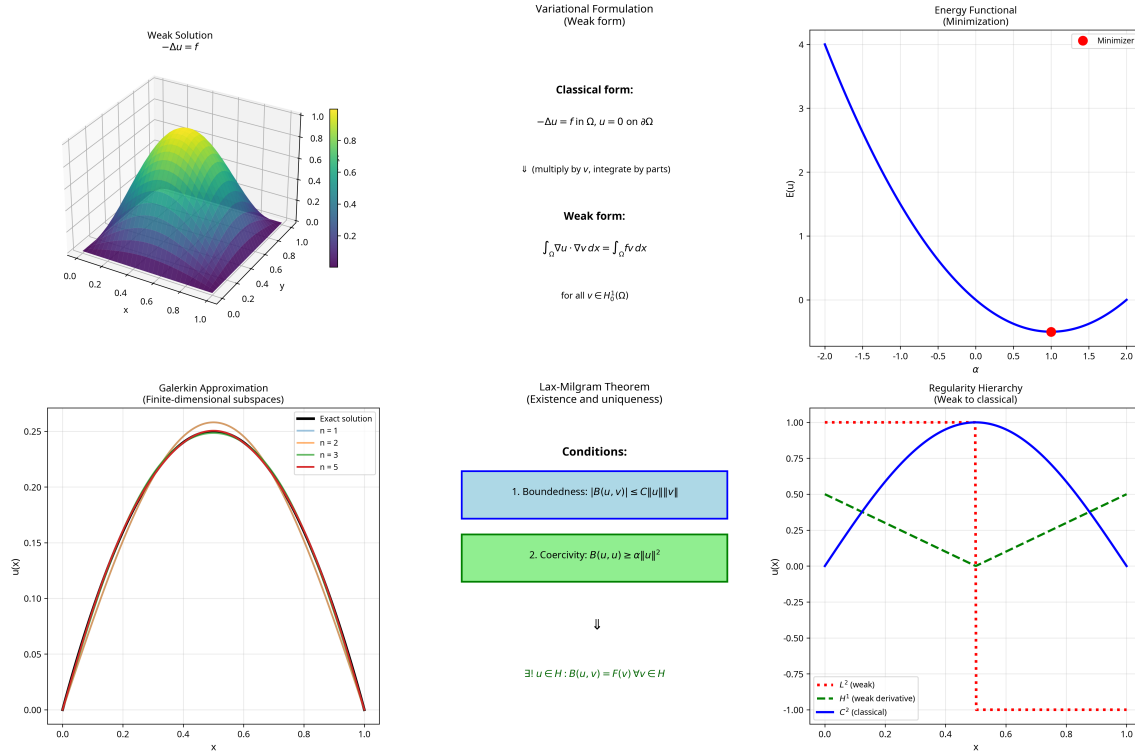


Figure 5: **Weak Solutions to Elliptic Equations.** This figure illustrates weak solutions. **(Top left)** Weak solution to the Poisson equation. **(Top middle)** Variational formulation showing transition from classical to weak form. **(Top right)** Energy functional and its minimizer. **(Bottom left)** Galerkin approximation using finite-dimensional subspaces. **(Bottom middle)** Lax-Milgram theorem conditions for existence and uniqueness. **(Bottom right)** Regularity hierarchy from weak to classical solutions.

## 5.1 Regularity of Weak Solutions

Once we have established the existence of a weak solution, a natural question is whether this solution is also a classical solution, i.e., whether it is smooth. This is the subject of **elliptic regularity theory**. The general principle is that if the data of the problem (the domain  $\Omega$ , the coefficients of the operator, and the right-hand side  $f$ ) are smooth, then the weak solution is also smooth.

**Theorem 5.2** (Interior Regularity). *Let  $u \in H^1(\Omega)$  be a weak solution of  $-\Delta u = f$ . If  $f \in H^k(\Omega)$  for some  $k \geq 0$ , then  $u \in H_{loc}^{k+2}(\Omega)$ . In particular, if  $f \in C^\infty(\Omega)$ , then  $u \in C^\infty(\Omega)$ .*

There are also regularity results up to the boundary, which state that if the boundary  $\partial\Omega$  is smooth, then the solution is smooth up to the boundary.

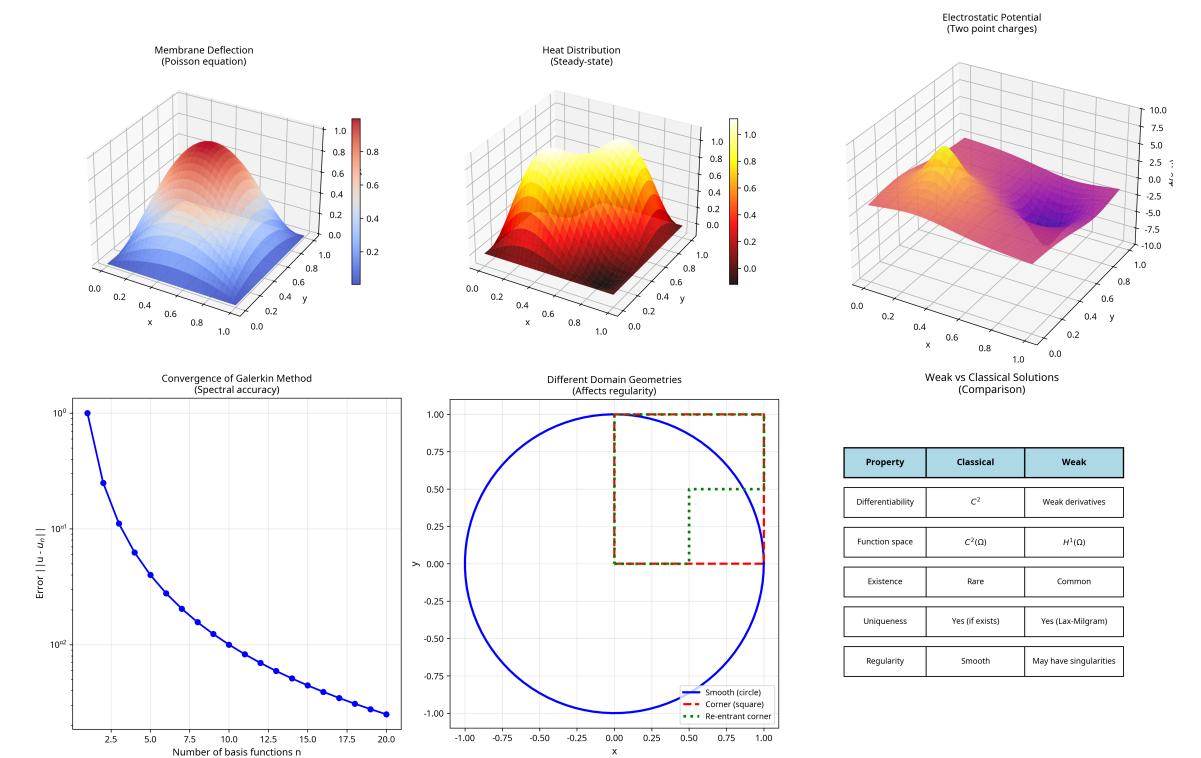


Figure 6: **Applications and Examples.** This figure shows applications of Sobolev space theory. **(Top left)** Membrane deflection modeled by the Poisson equation. **(Top middle)** Steady-state heat distribution. **(Top right)** Electrostatic potential from two point charges. **(Bottom left)** Convergence of the Galerkin method showing spectral accuracy. **(Bottom middle)** Different domain geometries affecting regularity. **(Bottom right)** Comparison table of weak versus classical solutions.

## 6 Conclusion

Sobolev spaces provide the essential functional analytic framework for the modern theory of partial differential equations. They allow us to define the concept of a weak solution, which is crucial for studying PDEs that do not have classical solutions. The Sobolev embedding theorems and the Rellich-Kondrachov compactness theorem are powerful tools for proving the existence of weak solutions. The Lax-Milgram theorem provides a general method for solving linear elliptic problems in a weak sense. Finally, elliptic regularity theory allows us to show that weak solutions are often classical solutions, thus bridging the gap between the modern and classical theories of PDEs. The concepts and techniques introduced in this chapter are fundamental for the study of more advanced topics in PDEs, including nonlinear equations and evolution problems.

## References

[1] Adams, R. A., & Fournier, J. J. F. (2003). *Sobolev Spaces* (2nd ed.). Academic Press.

[2] Evans, L. C. (2010). *Partial Differential Equations* (2nd ed.). American Mathematical Society.

[3] Brézis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer.

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- [4] Gilbarg, D., & Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. Springer.
  - [5] Lieb, E. H., & Loss, M. (2001). *Analysis* (2nd ed.). American Mathematical Society.

# Chapter 7

## Advanced Topics in Nonlinear Weak Solutions

### 1 Introduction to Advanced Topics in Nonlinear Weak Solutions

Building upon the foundations of Sobolev spaces and weak solutions established in the previous chapter, we now delve into more advanced techniques for tackling nonlinear partial differential equations. While the Lax-Milgram theorem provides a powerful tool for linear elliptic problems, many PDEs of interest in physics and engineering are inherently nonlinear. The analysis of such equations requires a more sophisticated functional analytic framework.

This chapter introduces several powerful methods for proving the existence of weak solutions to nonlinear PDEs. We will explore three main pillars of modern nonlinear analysis: monotone operator theory, variational methods for problems with critical growth, and bifurcation theory. These techniques have been instrumental in making progress on a wide range of challenging problems, from the study of non-Newtonian fluids to the analysis of geometric problems and pattern formation.

### 2 Monotone Operator Theory

A large class of nonlinear elliptic problems can be formulated in terms of finding a zero of a nonlinear operator  $A : X \rightarrow X^*$ , where  $X$  is a Banach space and  $X^*$  is its dual. The theory of **monotone operators**, developed in the 1960s by George Minty and Felix Browder, provides a general framework for solving such equations.

**Definition 2.1** (Monotone Operator). Let  $X$  be a real Banach space. An operator  $A : X \rightarrow X^*$  is said to be **monotone** if for all  $u, v \in X$ ,

$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad (7.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $X^*$  and  $X$ .

Monotonicity is a generalization of the property of being non-decreasing for real-valued functions. The main result of monotone operator theory is the **Browder-Minty theorem**, which gives conditions for a monotone operator to be surjective.

**Theorem 2.2** (Browder-Minty Theorem). *Let  $X$  be a reflexive, separable, real Banach space. Let  $A : X \rightarrow X^*$  be a monotone, coercive, and continuous operator. Then  $A$  is surjective, i.e., for every  $f \in X^*$ , there exists a solution  $u \in X$  to the equation  $A(u) = f$ .*

This theorem is a powerful tool for proving the existence of weak solutions to a wide class of nonlinear elliptic equations, including the p-Laplacian equation.

**Example 2.3** (The p-Laplacian). The p-Laplacian operator is defined by  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . The corresponding weak formulation leads to the operator  $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$  defined by

$$\langle A(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \quad (7.2)$$

This operator can be shown to be monotone, coercive, and continuous, and thus the Browder-Minty theorem guarantees the existence of a weak solution to the p-Laplace equation  $-\Delta_p u = f$  for any  $f \in W^{-1,p'}(\Omega)$ .

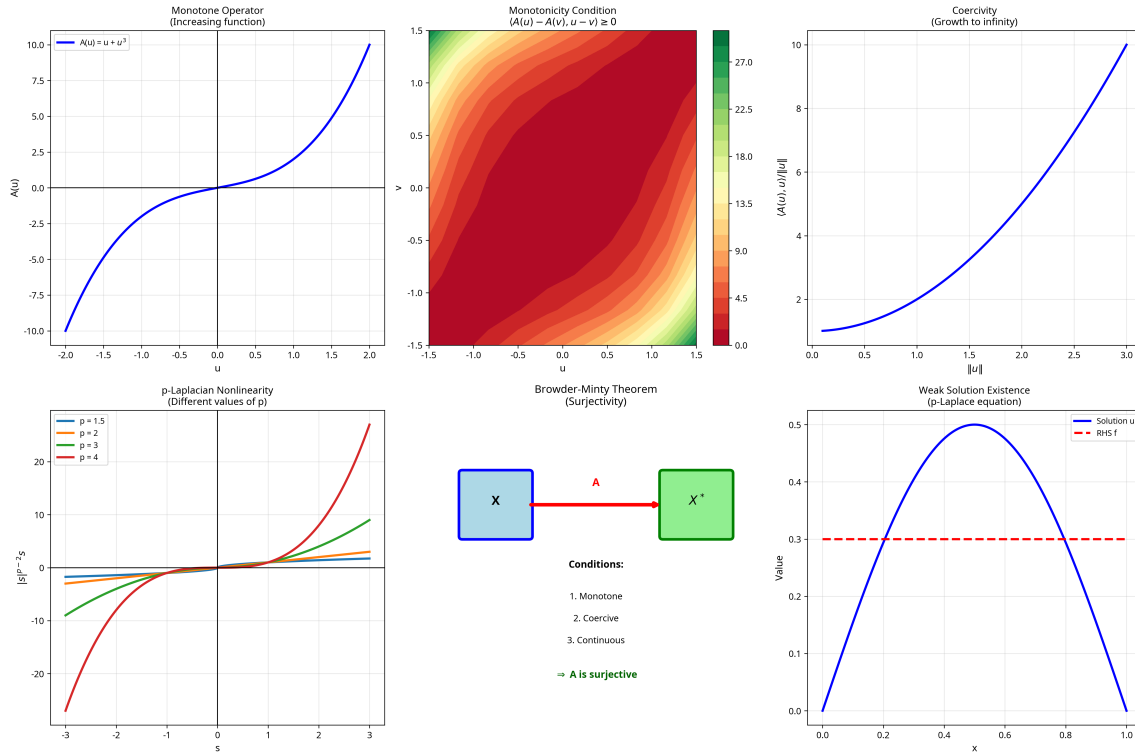


Figure 1: **Monotone Operators and the Browder-Minty Theorem.** This figure illustrates the theory of monotone operators. **(Top left)** Example of a monotone operator showing increasing behaviour. **(Top middle)** Visualization of the monotonicity condition as a contour plot. **(Top right)** Coercivity property showing growth to infinity. **(Bottom left)** The p-Laplacian nonlinearity for different values of  $p$ . **(Bottom middle)** Browder-Minty theorem diagram showing conditions for surjectivity. **(Bottom right)** Example of weak solution existence for the p-Laplace equation.

## 2.1 Proof of the Browder-Minty Theorem

The proof of the Browder-Minty theorem is a beautiful application of the Galerkin method and compactness arguments. We provide a detailed proof below.

*Proof.* Let  $f \in X^*$  be given. We want to find  $u \in X$  such that  $A(u) = f$ . The proof proceeds in several steps.

**Step 1: Galerkin Approximation.** Since  $X$  is a separable Banach space, there exists a countable basis  $\{w_1, w_2, \dots\}$  for  $X$ . For each  $n \in \mathbb{N}$ , we define the finite-dimensional subspace  $X_n = \text{span}\{w_1, \dots, w_n\}$ . We seek a solution  $u_n \in X_n$  to the projected equation

$$\langle A(u_n), w_j \rangle = \langle f, w_j \rangle \quad \text{for } j = 1, \dots, n \quad (7.3)$$

Let  $u_n = \sum_{i=1}^n c_i w_i$ . Then this is a system of  $n$  nonlinear equations in the  $n$  unknowns  $c_1, \dots, c_n$ . We define an operator  $P_n : X_n \rightarrow X_n$  by  $P_n(v) = \sum_{j=1}^n (\langle A(v) - f, w_j \rangle) w_j$ . A solution to the Galerkin system is a vector  $u_n$  such that  $P_n(u_n) = 0$ . We consider the inner product  $\langle P_n(v), v \rangle = \langle A(v) - f, v \rangle$ . By coercivity of  $A$ , for  $\|v\| = R$  large enough,  $\langle A(v), v \rangle > \|v\| \|f\|$ , so  $\langle P_n(v), v \rangle > 0$ . By Brouwer's fixed-point theorem, there exists a solution  $u_n \in X_n$  to the Galerkin system.

**Step 2: A Priori Estimates.** We now derive a uniform bound on the sequence of approximate solutions  $(u_n)$ . From the Galerkin equation, we have  $\langle A(u_n), u_n \rangle = \langle f, u_n \rangle$ . By the coercivity of  $A$ , there exists a constant  $c > 0$  such that  $\langle A(u_n), u_n \rangle \geq c\|u_n\|^p$ . Thus,  $c\|u_n\|^p \leq \|f\|\|u_n\|$ , which implies that  $\|u_n\|$  is bounded independently of  $n$ .

**Step 3: Compactness and Weak Convergence.** Since  $X$  is a reflexive Banach space, every bounded sequence has a weakly convergent subsequence. Thus, there exists a subsequence, which we still denote by  $(u_n)$ , and an element  $u \in X$  such that  $u_n \rightharpoonup u$  weakly in  $X$ . Since  $A$  is a continuous operator,  $A(u_n)$  is a bounded sequence in  $X^*$ . Since  $X^*$  is also reflexive, there exists a subsequence of  $A(u_n)$  that converges weakly to some  $\chi \in X^*$ .

**Step 4: Passing to the Limit.** The final step is to show that  $\chi = A(u)$  and that  $u$  is a solution to the original equation. This is the most technical part of the proof. By monotonicity of  $A$ , for any  $v \in X_k$  and  $n \geq k$ , we have  $\langle A(u_n) - A(v), u_n - v \rangle \geq 0$ . Taking the limit as  $n \rightarrow \infty$ , we get  $\langle \chi - A(v), u - v \rangle \geq 0$ . Now, let  $v = u - tw$  for  $w \in X$  and  $t > 0$ . Then  $\langle \chi - A(u - tw), tw \rangle \geq 0$ , which implies  $\langle \chi - A(u - tw), w \rangle \geq 0$ . Taking the limit as  $t \rightarrow 0$ , by the continuity of  $A$ , we get  $\langle \chi - A(u), w \rangle \geq 0$  for all  $w \in X$ . This implies  $\chi = A(u)$ .

Finally, we show that  $A(u) = f$ . For any fixed  $k$ , and for  $n \geq k$ , we have  $\langle A(u_n), w_k \rangle = \langle f, w_k \rangle$ . Taking the limit as  $n \rightarrow \infty$ , we get  $\langle A(u), w_k \rangle = \langle f, w_k \rangle$  for all  $k$ . Since the set  $\{w_k\}$  is a basis for  $X$ , this implies that  $A(u) = f$ .  $\square$

### 3 Variational Methods for Critical Growth Problems

Another powerful technique for solving nonlinear PDEs is the **calculus of variations**. The idea is to find a solution as a critical point (e.g., a minimizer) of an associated energy functional. For this to work, we need the functional to be coercive and weakly lower semicontinuous, which is guaranteed by the Rellich-Kondrachov theorem for subcritical problems.

However, a major difficulty arises in the case of **critical growth**, where the nonlinearity has the same growth rate as the one in the Sobolev embedding theorem. In this case, the embedding is not compact, and the weak lower semicontinuity of the energy functional is lost. This is a significant challenge, as many important problems in geometry and physics exhibit critical growth.

**Example 3.1** (The Yamabe Problem). A famous example is the Yamabe problem in differential geometry, which involves solving the equation

$$-\Delta u + c(x)u = f(x)u^{p-1} \quad (7.4)$$

on a compact Riemannian manifold, where  $p = \frac{2n}{n-2}$  is the critical Sobolev exponent. The solution of this equation allows one to find a metric of constant scalar curvature.

To overcome the lack of compactness, more sophisticated techniques are needed, such as the **concentration-compactness principle** of Pierre-Louis Lions, or methods based on the **mountain pass theorem** of Ambrosetti and Rabinowitz.

**Theorem 3.2** (Mountain Pass Theorem). *Let  $E$  be a real Banach space and let  $I \in C^1(E, \mathbb{R})$ . Suppose that  $I(0) = 0$  and there exist  $\rho, \alpha > 0$  such that  $I(u) \geq \alpha$  for all  $u \in E$  with  $\|u\| = \rho$ , and there exists  $e \in E$  with  $\|e\| > \rho$  such that  $I(e) < 0$ . Then there exists a sequence  $(u_k)$  in  $E$  such that  $I(u_k) \rightarrow c > 0$  and  $I'(u_k) \rightarrow 0$ , where  $c$  is the mountain pass level.*

Under certain additional conditions (the Palais-Smale condition), one can show that this sequence converges to a critical point of  $I$ , which is a weak solution of the corresponding PDE.



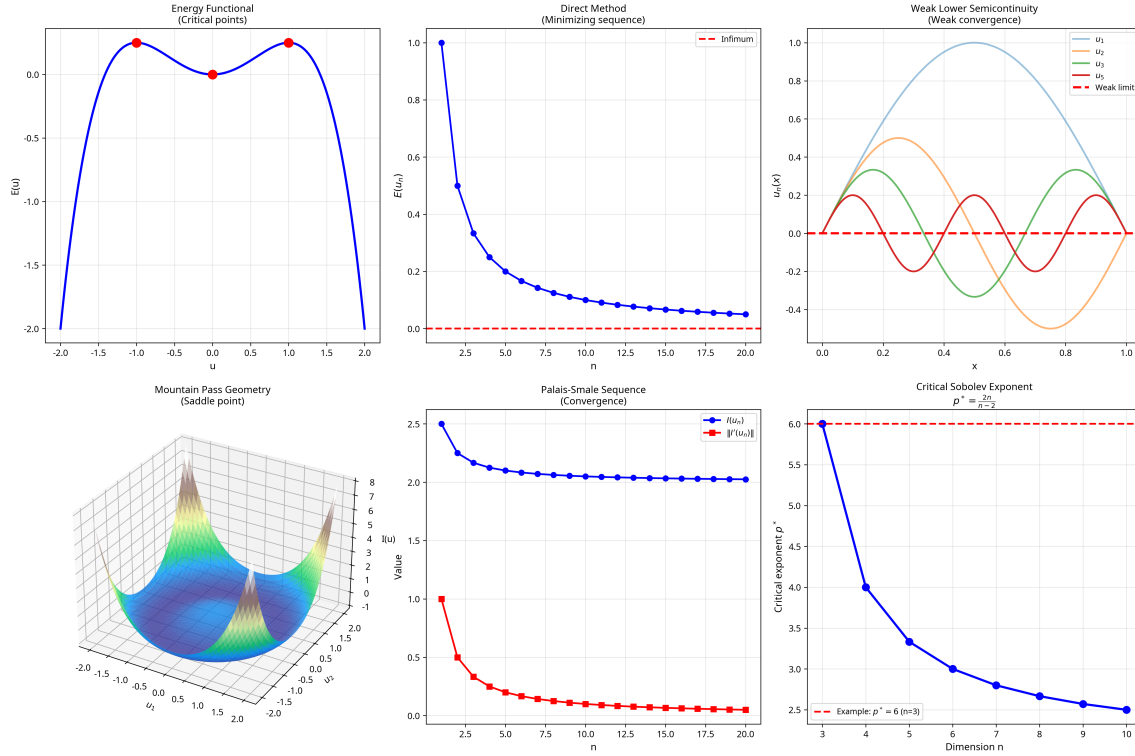


Figure 2: **Variational Methods and Critical Points.** This figure illustrates variational methods for nonlinear PDEs. **(Top left)** Energy functional with multiple critical points. **(Top middle)** Direct method showing a minimizing sequence converging to the infimum. **(Top right)** Weak lower semicontinuity and weak convergence. **(Bottom left)** Mountain pass geometry showing saddle point structure. **(Bottom middle)** Palais-Smale sequence with energy converging and gradient vanishing. **(Bottom right)** Critical Sobolev exponent as a function of dimension.

### 3.1 The Palais-Smale Condition

The Palais-Smale (PS) condition is a compactness condition that is crucial for the application of variational methods. It ensures that any sequence that is 'almost' a critical point (in the sense that the derivative of the functional is small) has a convergent subsequence. This prevents the 'loss of mass' at infinity that can occur in problems with a lack of compactness.

**Definition 3.3** (Palais-Smale Condition). Let  $I : E \rightarrow \mathbb{R}$  be a  $C^1$  functional on a Banach space  $E$ . We say that  $I$  satisfies the **Palais-Smale (PS) condition** if any sequence  $(u_k)$  in  $E$  such that  $I(u_k)$  is bounded and  $I'(u_k) \rightarrow 0$  in  $E^*$  has a convergent subsequence.

Verifying the PS condition is often the most difficult part of applying the mountain pass theorem. For problems with critical growth, the PS condition may fail at certain energy levels. This is where the concentration-compactness principle comes into play, as it provides a precise description of how the PS condition can fail.

### 3.2 Concentration-Compactness Principle

The concentration-compactness principle, developed by Pierre-Louis Lions, is a powerful tool for analyzing sequences in Sobolev spaces that lack compactness. It provides a dichotomy for weakly convergent sequences, stating that such a sequence either converges strongly, or it splits into a sum of 'bubbles' that concentrate at different points in the domain.

This principle allows one to analyze the behavior of minimizing sequences for variational problems with critical growth and to prove the existence of solutions by ruling out the possibility of concentration.

## 4 Bifurcation Theory

**Bifurcation theory** is the study of how the set of solutions to a nonlinear equation changes as a parameter in the equation is varied. A bifurcation occurs when a small change in the parameter leads to a qualitative change in the solution set, such as the appearance of new solutions.

Consider a nonlinear equation of the form

$$F(u, \lambda) = 0 \tag{7.5}$$

where  $u$  is the solution and  $\lambda$  is a real parameter. We are interested in how the solutions  $u$  depend on  $\lambda$ . A bifurcation point is a point  $(\lambda_0, u_0)$  such that in any neighborhood of this point, there are solutions that are not on the same solution branch.

**Example 4.1** (Bifurcation from a simple eigenvalue). A classical example is the problem

$$-\Delta u = \lambda u + u^3 \tag{7.6}$$

in a bounded domain  $\Omega$  with zero boundary conditions. This equation always has the trivial solution  $u = 0$ . However, as  $\lambda$  increases past the first eigenvalue of the Laplacian, two new non-trivial solutions bifurcate from the trivial solution branch.

The study of bifurcation phenomena often involves techniques from degree theory, such as the **Leray-Schauder degree**, and methods based on the implicit function theorem and the Lyapunov-Schmidt reduction.

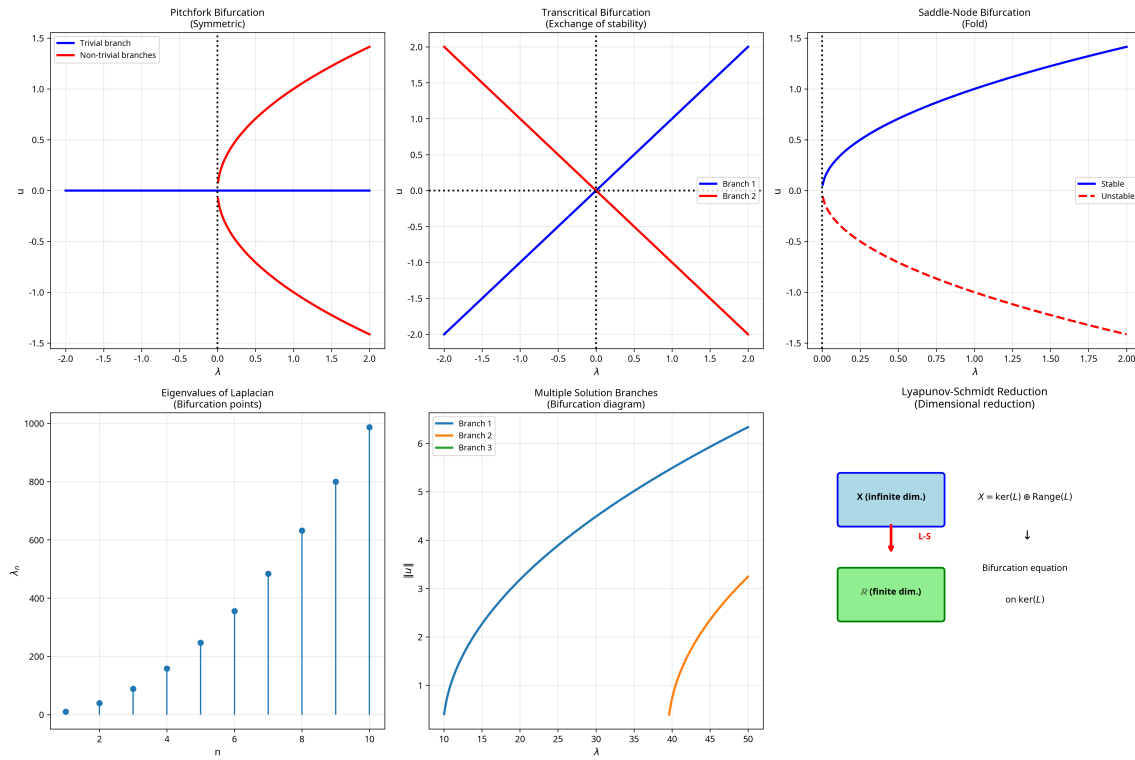


Figure 3: **Bifurcation Theory.** This figure illustrates bifurcation phenomena in nonlinear PDEs. **(Top left)** Pitchfork bifurcation showing symmetric branching. **(Top middle)** Transcritical bifurcation with exchange of stability. **(Top right)** Saddle-node bifurcation (fold). **(Bottom left)** Eigenvalues of the Laplacian as bifurcation points. **(Bottom middle)** Multiple solution branches in a bifurcation diagram. **(Bottom right)** Lyapunov-Schmidt reduction showing dimensional reduction from infinite to finite dimensions.

#### 4.1 Leray-Schauder Degree

The Leray-Schauder degree is a topological invariant that generalizes the winding number of a curve in the complex plane to infinite-dimensional spaces. It provides a way to count the number of solutions to a nonlinear equation inside a given domain.

If the Leray-Schauder degree of an operator is non-zero on a certain domain, then there must be at least one solution to the equation inside that domain. This is a powerful tool for proving the existence of solutions and for studying how the solution set changes as parameters are varied.

#### 4.2 Lyapunov-Schmidt Reduction

The Lyapunov-Schmidt reduction is a technique for reducing an infinite-dimensional bifurcation problem to a finite-dimensional one. The idea is to split the space into the kernel of the linearized operator and its complement. The original equation is then split into two coupled equations, one on the kernel and one on its complement. The equation on the complement can be solved using the implicit function theorem, and the problem is reduced to solving a finite-dimensional bifurcation equation on the kernel.

This technique is particularly useful for studying bifurcation from a simple eigenvalue, where the kernel of the linearized operator is one-dimensional.

## 5 Regularity of Nonlinear Weak Solutions

As in the linear case, once we have established the existence of a weak solution to a nonlinear PDE, we are interested in its regularity. The regularity theory for nonlinear equations is much more complex than for linear equations. The regularity of the solution depends not only on the regularity of the data but also on the structure of the nonlinearity.

For semilinear equations of the form  $-\Delta u = f(u)$ , a bootstrapping argument can often be used to show that if  $f$  is smooth, then any weak solution is also smooth. However, for quasilinear equations like the  $p$ -Laplacian, the regularity theory is much more challenging. The solutions are typically not  $C^2$  but only  $C^{1,\alpha}$  for some  $\alpha > 0$ . The study of the regularity of solutions to the Navier-Stokes equations is one of the most famous open problems in mathematics (one of the Millennium Prize Problems).

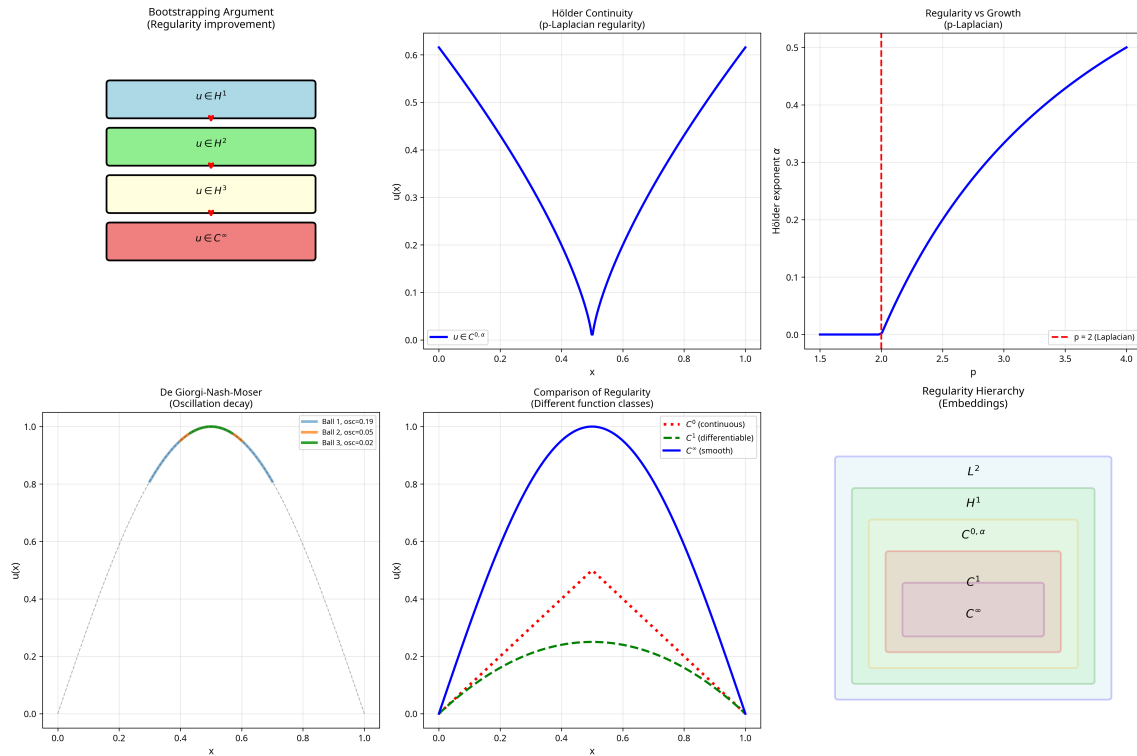


Figure 4: **Regularity Theory for Nonlinear Equations.** This figure illustrates regularity concepts. **(Top left)** Bootstrapping argument showing progressive regularity improvement. **(Top middle)** Hölder continuous function typical of  $p$ -Laplacian solutions. **(Top right)** Regularity versus growth parameter  $p$ . **(Bottom left)** De Giorgi-Nash-Moser theory showing oscillation decay. **(Bottom middle)** Comparison of different regularity levels. **(Bottom right)** Regularity hierarchy showing nested function spaces.

### 5.1 De Giorgi-Nash-Moser Theory

For linear elliptic equations with rough coefficients, the De Giorgi-Nash-Moser theory provides a way to prove the Hölder continuity of weak solutions. This theory has been extended to certain classes of quasilinear equations, such as the  $p$ -Laplacian, and is a fundamental tool in the regularity theory for nonlinear PDEs.

## 6 Variational Inequalities and Obstacle Problems

An important class of nonlinear problems arises from **variational inequalities**, where the solution is required to satisfy an inequality rather than an equation. A classical example is the **obstacle problem**, where we seek a function  $u$  that minimizes an energy functional subject to the constraint that  $u \geq \psi$  for some given obstacle function  $\psi$ .

The weak formulation of the obstacle problem is: find  $u \in K$  such that

$$\int_{\Omega} \nabla u \cdot \nabla (v - u) \, dx \geq \int_{\Omega} f(v - u) \, dx \quad (7.7)$$

for all  $v \in K$ , where  $K = \{v \in H_0^1(\Omega) : v \geq \psi\}$  is a closed convex set.

The solution to this problem can be characterized as the projection of the unconstrained solution onto the convex set  $K$ . The set where  $u = \psi$  is called the **contact set**, and its boundary is called the **free boundary**. The study of free boundary problems is an active area of research in PDE theory.

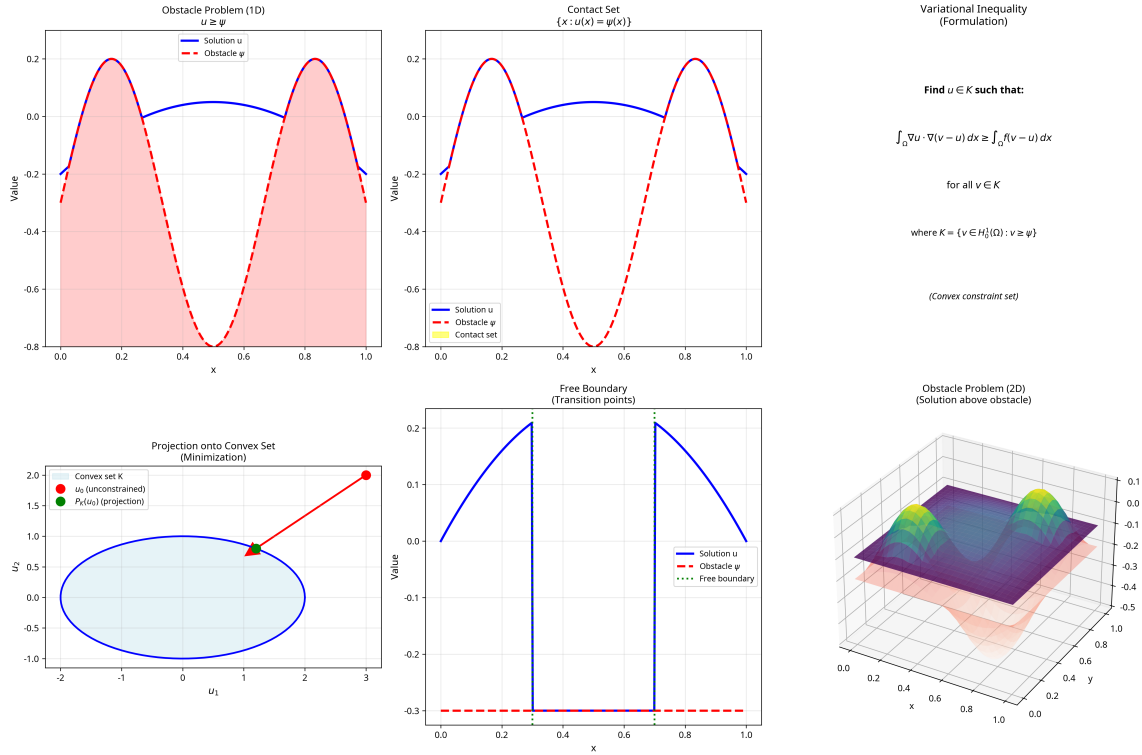


Figure 5: **Variational Inequalities and Obstacle Problems.** This figure illustrates variational inequalities. **(Top left)** One-dimensional obstacle problem showing solution staying above obstacle. **(Top middle)** Contact set where solution touches the obstacle. **(Top right)** Variational inequality formulation. **(Bottom left)** Projection onto convex set in finite dimensions. **(Bottom middle)** Free boundary separating contact and non-contact regions. **(Bottom right)** Two-dimensional obstacle problem visualization.

## 7 Applications

The techniques developed in this chapter have numerous applications across mathematics, physics, and engineering. We will now explore some of these applications in more detail.

### 7.1 Non-Newtonian Fluids and the p-Laplacian

Many fluids, such as ketchup, toothpaste, and blood, do not follow the linear stress-strain relationship of Newtonian fluids. These **non-Newtonian fluids** are often modeled by a power-law relationship, where the stress tensor  $\tau$  is related to the rate of strain tensor  $D$  by  $\tau = \mu|D|^{p-2}D$ , where  $p$  is a parameter that characterizes the fluid. For  $p = 2$ , we recover the Newtonian case. For  $p < 2$ , the fluid is shear-thinning, and for  $p > 2$ , it is shear-thickening.

The stationary, incompressible flow of such a fluid is governed by the equations

$$-\operatorname{div}(\mu|D(u)|^{p-2}D(u)) + \nabla\pi = f \quad (7.8)$$

$$\operatorname{div}(u) = 0 \quad (7.9)$$

where  $u$  is the velocity field and  $\pi$  is the pressure. The first equation is a generalization of the Stokes equations and involves the p-Laplacian operator. The existence of weak solutions to this system can be established using the theory of monotone operators.

### 7.2 Minimal Surfaces

A classical problem in the calculus of variations is to find a surface of minimal area with a given boundary. This leads to the **minimal surface equation**, which is a quasilinear elliptic PDE. For a surface given by the graph of a function  $u(x, y)$ , the area is given by the functional

$$I(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx \quad (7.10)$$

The Euler-Lagrange equation for this functional is

$$\operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad (7.11)$$

This is a highly nonlinear equation, and the existence of solutions requires advanced techniques from geometric measure theory and the calculus of variations.

### 7.3 Reaction-Diffusion Systems and Pattern Formation

Reaction-diffusion systems are mathematical models that describe how the concentration of one or more substances distributed in space changes under the influence of two processes: local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space. A simple example is the Fisher-KPP equation

$$u_t = D\Delta u + ru(1 - u) \quad (7.12)$$

which models the spread of an advantageous gene in a population. The interplay between the reaction and diffusion terms can lead to the formation of complex spatial patterns, such as traveling waves, spiral waves, and Turing patterns. Bifurcation theory is a key tool for understanding how these patterns arise as parameters in the system are varied.

### 7.4 Elasticity and Contact Problems

In solid mechanics, variational inequalities are used to model contact problems in elasticity. For example, consider an elastic membrane stretched over a domain  $\Omega$  and constrained to lie above an obstacle. The displacement of the membrane minimizes the elastic energy subject to the obstacle constraint. This leads to a variational inequality, and the solution exhibits a free boundary that separates the region where the membrane is in contact with the obstacle from the region where it is not.

## 7.5 Optimal Control

Variational inequalities also arise in the field of optimal control, where one seeks to find a control that minimizes a certain cost functional subject to constraints on the state and control variables. The necessary conditions for optimality often take the form of a variational inequality.

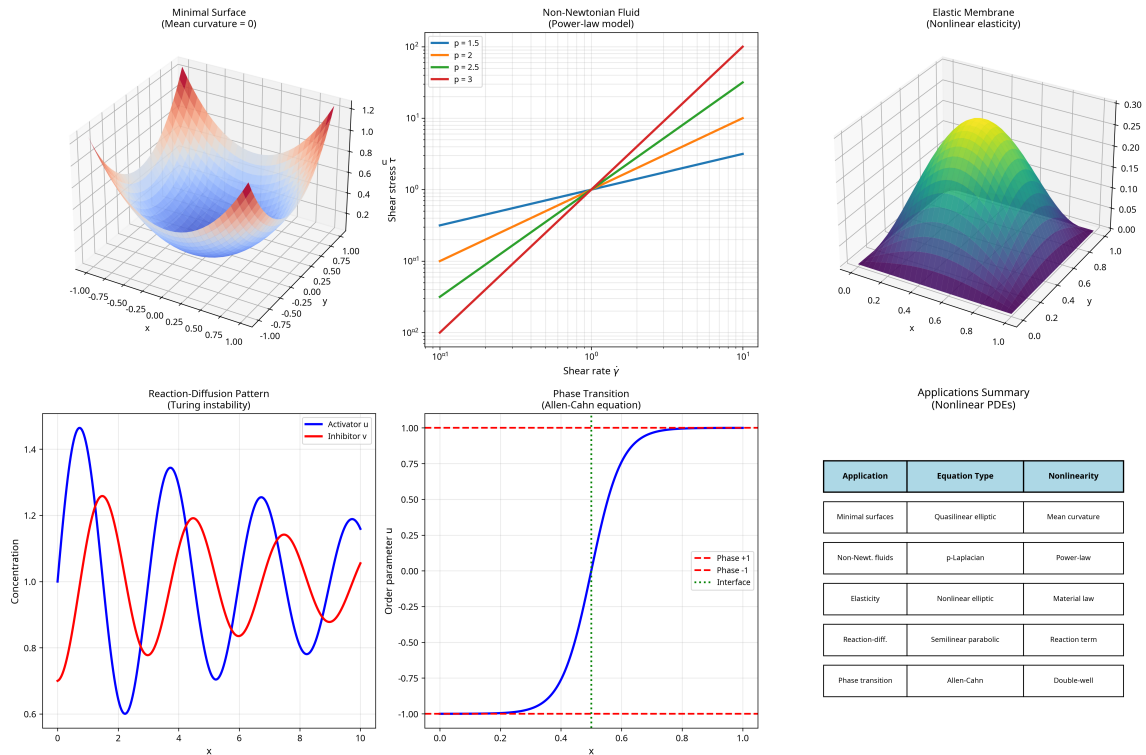


Figure 6: **Applications of Nonlinear Weak Solutions.** This figure shows applications across different fields. (**Top left**) Minimal surface with zero mean curvature. (**Top middle**) Non-Newtonian fluid showing power-law behaviour. (**Top right**) Elastic membrane under nonlinear elasticity. (**Bottom left**) Reaction-diffusion pattern from Turing instability. (**Bottom middle**) Phase transition modeled by Allen-Cahn equation. (**Bottom right**) Summary table of applications and their corresponding nonlinear PDEs.

## 8 Conclusion

This chapter has provided a glimpse into the rich and complex world of nonlinear partial differential equations. We have seen how the functional analytic tools developed in the previous chapters can be extended and adapted to tackle nonlinear problems. Monotone operator theory, variational methods, and bifurcation theory are just a few of the many powerful techniques that have been developed to study nonlinear PDEs. The field is still very active, with many fundamental questions remaining open, particularly in the areas of regularity and the behavior of solutions to equations arising from fluid dynamics and other areas of physics.

## References

- [1] Browder, F. E. (1963). Nonlinear elliptic boundary value problems. *Bulletin of the American Mathematical Society*, 69(6), 862-874.
- [2] Lions, P. L. (1984). The concentration-compactness principle in the calculus of variations. The locally compact case, part 1. *Annales de l'Institut Henri Poincaré C*, 1(2), 109-145.

- [3] Ambrosetti, A., & Rabinowitz, P. H. (1973). Dual variational methods in critical point theory and applications. *Journal of Functional Analysis*, 14(4), 349-381.
- [4] Crandall, M. G., & Rabinowitz, P. H. (1971). Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8(2), 321-340.
- [5] Kinderlehrer, D., & Stampacchia, G. (2000). *An Introduction to Variational Inequalities and Their Applications*. SIAM.



# Chapter 8

## Stochastic Partial Differential Equations

### 1 Introduction to Stochastic Partial Differential Equations

Stochastic Partial Differential Equations (SPDEs) represent one of the most challenging and rapidly developing areas of modern mathematical analysis. They arise naturally when one seeks to model physical, biological, or financial systems that evolve in both space and time under the influence of random perturbations. Unlike ordinary stochastic differential equations (SDEs), which describe the evolution of finite-dimensional random processes, SPDEs govern the dynamics of infinite-dimensional stochastic processes, such as random fields.

The mathematical theory of SPDEs is considerably more intricate than that of deterministic PDEs. This complexity stems from the need to combine tools from functional analysis, operator theory, and probability theory in a coherent framework. A central difficulty lies in giving precise mathematical meaning to the noise term, which is often modeled as space-time white noise—a highly singular object that is neither a function nor even a measure in the classical sense.

#### 1.1 Historical Development

The systematic study of SPDEs began in the 1960s and 1970s, building upon the foundational work of Kiyosi Itô on stochastic calculus. Walsh’s influential 1986 monograph provided the first comprehensive treatment of SPDEs driven by space-time white noise. Subsequent developments by Da Prato and Zabczyk in the 1990s established the semigroup approach as a powerful tool for analyzing linear and semilinear SPDEs in Hilbert spaces.

In recent years, the field has witnessed remarkable breakthroughs. Martin Hairer’s theory of regularity structures, introduced in 2014, provided a general framework for constructing solutions to highly singular SPDEs, including the stochastic quantization equations arising in quantum field theory. This work was recognized with the Fields Medal in 2014.

#### 1.2 Applications

SPDEs arise in a diverse array of applications across the natural sciences, engineering, and finance. In statistical mechanics, they describe the evolution of systems with many degrees of freedom subject to thermal fluctuations. In fluid dynamics, stochastic Navier-Stokes equations model turbulent flow. In mathematical finance, SPDEs govern the dynamics of interest rate curves and volatility surfaces. In neuroscience, they describe the propagation of electrical signals in dendrites subject to ion channel noise.

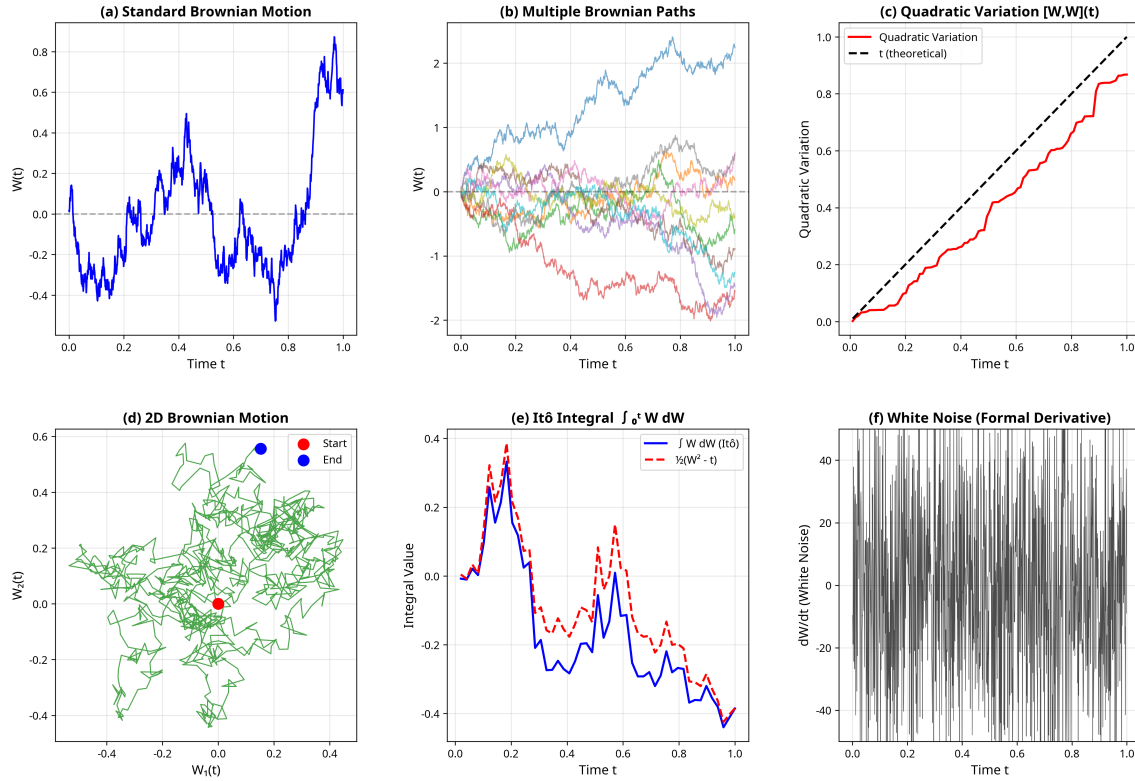


Figure 1: **Brownian Motion and Stochastic Processes.** This figure illustrates fundamental concepts in stochastic calculus. (a) A single path of standard Brownian motion showing continuous but nowhere differentiable behaviour. (b) Multiple independent Brownian paths demonstrating the variability of stochastic processes. (c) Quadratic variation  $[W, W](t)$  converging to  $t$ , a key property distinguishing Brownian motion from smooth functions. (d) Two-dimensional Brownian motion as a projection of cylindrical Brownian motion in infinite dimensions. (e) Itô integral  $\int_0^t W dW = \frac{1}{2}(W^2 - t)$  demonstrating the correction term from Itô's formula. (f) White noise as the formal derivative of Brownian motion, showing highly irregular behaviour.

## 2 Stochastic Calculus in Infinite Dimensions

To develop a rigorous theory of SPDEs, we must first extend the machinery of stochastic calculus from finite-dimensional spaces to infinite-dimensional Hilbert spaces. This extension is non-trivial and requires careful attention to topological and measure-theoretic subtleties.

### 2.1 Cylindrical Brownian Motion

Let  $H$  be a separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$  and norm  $\| \cdot \|_H$ . In the finite-dimensional case, Brownian motion can be characterized by its covariance structure. In infinite dimensions, we encounter an immediate difficulty: there does not exist a Gaussian measure on  $H$  with identity covariance operator unless  $H$  is finite-dimensional. This is a consequence of the fact that the trace of the identity operator on an infinite-dimensional space is infinite.

To circumvent this obstacle, we introduce the notion of cylindrical Brownian motion, which is not a genuine  $H$ -valued process but rather a family of real-valued Brownian motions indexed by elements of  $H$ .

**Definition 2.1** (Cylindrical Brownian Motion). Let  $H$  be a separable Hilbert space. A **cylindrical Brownian motion** on  $H$  is a family of real-valued Brownian motions  $\{W_h(t)\}_{h \in H, t \geq 0}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that:

- (i) For each  $h \in H$ , the process  $W_h(t)$  is a standard real-valued Brownian motion.
- (ii) The mapping  $h \mapsto W_h(t)$  is linear for each fixed  $t \geq 0$ .
- (iii) For any  $h_1, h_2 \in H$  and  $s, t \geq 0$ , the covariance is given by

$$\mathbb{E}[W_{h_1}(t)W_{h_2}(s)] = (t \wedge s)\langle h_1, h_2 \rangle_H. \quad (8.1)$$

This definition captures the essential property that the covariance structure is determined by the inner product on  $H$ . However, for a general separable Hilbert space  $H$ , the cylindrical Brownian motion does not take values in  $H$  itself. To obtain an  $H$ -valued process, we must introduce a covariance operator.

## 2.2 $Q$ -Wiener Processes

Let  $Q : H \rightarrow H$  be a positive, self-adjoint, trace-class operator. Such an operator can be diagonalized with respect to an orthonormal basis  $\{e_k\}_{k=1}^\infty$  of  $H$ :

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, 3, \dots \quad (8.2)$$

where  $\lambda_k \geq 0$  are the eigenvalues and  $\sum_{k=1}^\infty \lambda_k = \text{Tr}(Q) < \infty$ .

**Definition 2.2** ( $Q$ -Wiener Process). A  **$Q$ -Wiener process** on  $H$  is an  $H$ -valued stochastic process  $W_Q(t)$  defined by

$$W_Q(t) = \sum_{k=1}^\infty \sqrt{\lambda_k} \beta_k(t) e_k \quad (8.3)$$

where  $\{\beta_k(t)\}_{k=1}^\infty$  are independent standard real-valued Brownian motions.

The series in equation (8.3) converges in  $L^2(\Omega; H)$  because

$$\mathbb{E}[\|W_Q(t)\|_H^2] = \sum_{k=1}^\infty \lambda_k \mathbb{E}[\beta_k(t)^2] = t \sum_{k=1}^\infty \lambda_k = t \text{Tr}(Q) < \infty. \quad (8.4)$$

The  $Q$ -Wiener process is the infinite-dimensional analogue of Brownian motion with covariance matrix  $Q$ . When  $Q = I$  (the identity operator), we recover the cylindrical Brownian motion, which does not take values in  $H$  unless  $H$  is finite-dimensional.

## 2.3 Stochastic Integration

The construction of the stochastic integral with respect to a cylindrical Brownian motion or a  $Q$ -Wiener process is a delicate matter. We must specify the class of integrands for which the integral is well-defined.

Let  $L(H)$  denote the space of bounded linear operators on  $H$ , and let  $L_2(H)$  denote the space of Hilbert-Schmidt operators on  $H$ . An operator  $T \in L(H)$  is Hilbert-Schmidt if

$$\|T\|_{L_2(H)}^2 = \sum_{k=1}^\infty \|Te_k\|_H^2 < \infty \quad (8.5)$$

for any orthonormal basis  $\{e_k\}_{k=1}^\infty$  of  $H$ .

**Definition 2.3** (Stochastic Integral). Let  $\Phi : [0, T] \times \Omega \rightarrow L_2(H)$  be a predictable process such that

$$\mathbb{E} \left[ \int_0^T \|\Phi(s)\|_{L_2(H)}^2 ds \right] < \infty. \quad (8.6)$$

The **stochastic integral** of  $\Phi$  with respect to the cylindrical Brownian motion  $W$  is defined by

$$\int_0^t \Phi(s) dW(s) = \sum_{k=1}^{\infty} \int_0^t \Phi(s) e_k d\beta_k(s) \quad (8.7)$$

where the series converges in  $L^2(\Omega; H)$ .

This definition extends the Itô integral from finite to infinite dimensions. The key requirement is that  $\Phi$  be Hilbert-Schmidt, which ensures that the sum over modes converges.

## 2.4 Itô's Formula in Infinite Dimensions

Itô's formula is the cornerstone of stochastic calculus, providing a chain rule for stochastic processes. In infinite dimensions, the formula takes a more complex form due to the presence of the trace term.

**Theorem 2.4** (Itô's Formula in Infinite Dimensions). *Let  $u(t)$  be the mild solution of the stochastic evolution equation*

$$du(t) = (Au(t) + F(t, u(t)))dt + B(t, u(t))dW(t) \quad (8.8)$$

where  $A$  is the generator of a  $C_0$ -semigroup on  $H$ ,  $F : [0, T] \times H \rightarrow H$  and  $B : [0, T] \times H \rightarrow L_2(H)$  are suitable coefficients. Let  $\Phi : H \rightarrow \mathbb{R}$  be a  $C^2$  function with bounded first and second derivatives. Then

$$\begin{aligned} d\Phi(u(t)) = & \left( \langle \Phi'(u(t)), Au(t) + F(t, u(t)) \rangle_H + \frac{1}{2} \text{Tr}(\Phi''(u(t))B(t, u(t))QB(t, u(t))^*) \right) dt \\ & + \langle \Phi'(u(t)), B(t, u(t))dW(t) \rangle_H \end{aligned} \quad (8.9)$$

where  $\Phi'(u)$  denotes the Fréchet derivative of  $\Phi$  at  $u$ ,  $\Phi''(u)$  denotes the second Fréchet derivative, and  $Q$  is the covariance operator of the noise.

The proof of this theorem follows the same general strategy as in finite dimensions, using a Taylor expansion and the quadratic variation of the Brownian motion. However, the infinite-dimensional setting requires careful treatment of the trace term, which may be infinite if the covariance operator  $Q$  is not trace-class.

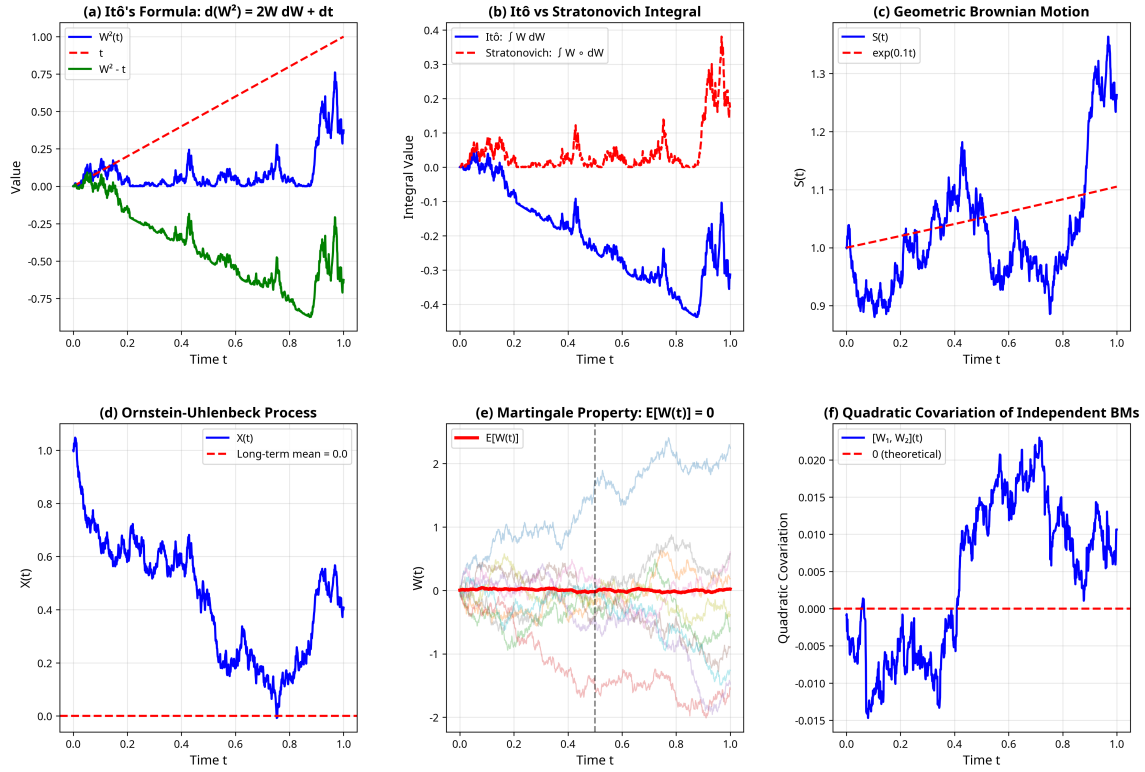


Figure 2: **Itô's Formula and Stochastic Calculus.** This figure illustrates key concepts in stochastic calculus. (a) Itô's formula for  $W^2$  showing that  $d(W^2) = 2W dW + dt$ , with the quadratic variation term  $t$  appearing as a correction. (b) Comparison of Itô and Stratonovich integrals, differing by the quadratic variation term. (c) Geometric Brownian motion  $dS = \mu S dt + \sigma S dW$  used in the Black-Scholes model. (d) Ornstein-Uhlenbeck process exhibiting mean reversion. (e) Martingale property of Brownian motion with  $\mathbb{E}[W(t)] = 0$ . (f) Quadratic covariation of independent Brownian motions vanishing in expectation.

### 3 The Stochastic Heat Equation

We now turn to the study of specific SPDEs, beginning with the stochastic heat equation, which serves as a prototypical example.

#### 3.1 Formulation of the Problem

Consider a bounded domain  $\Omega \subset \mathbb{R}^d$  with smooth boundary  $\partial\Omega$ . The **stochastic heat equation** is given by

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \dot{W}(t, x), & t > 0, x \in \Omega, \\ u(t, x) = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \quad (8.10)$$

where  $\dot{W}(t, x)$  denotes space-time white noise and  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function.

The notation  $\dot{W}(t, x)$  is purely formal, as white noise is not a function but a generalized random field. To give precise meaning to equation (8.10), we reformulate it as a stochastic evolution equation in the Hilbert space  $H = L^2(\Omega)$ .

### 3.2 Semigroup Formulation

Let  $A = \frac{1}{2}\Delta$  be the Laplacian with Dirichlet boundary conditions, viewed as an unbounded operator on  $L^2(\Omega)$  with domain  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ . The operator  $A$  generates a strongly continuous semigroup  $S(t) = e^{tA}$  on  $L^2(\Omega)$ , known as the heat semigroup.

We rewrite equation (8.10) as

$$du(t) = Au(t)dt + \sigma(u(t))dW(t) \quad (8.11)$$

where  $W(t)$  is a cylindrical Brownian motion on  $L^2(\Omega)$ .

A **mild solution** of equation (8.11) is a process  $u(t)$  satisfying

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\sigma(u(s))dW(s). \quad (8.12)$$

This integral formulation avoids the need to differentiate the solution, which may not be differentiable in the classical sense.

### 3.3 Existence and Uniqueness

The existence and uniqueness of mild solutions to the stochastic heat equation can be established using a fixed-point argument.

**Theorem 3.1** (Existence and Uniqueness for the Stochastic Heat Equation). *Let  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz continuous with Lipschitz constant  $L$ . Assume that  $u_0 \in L^2(\Omega)$ . Then there exists a unique mild solution  $u(t)$  to equation (8.11) in the space  $C([0, T]; L^2(\Omega))$  for any  $T > 0$ .*

*Proof Sketch.* We use a contraction mapping argument. Define the map  $\Gamma : C([0, T]; L^2(\Omega)) \rightarrow C([0, T]; L^2(\Omega))$  by

$$(\Gamma v)(t) = S(t)u_0 + \int_0^t S(t-s)\sigma(v(s))dW(s). \quad (8.13)$$

We must show that  $\Gamma$  is a contraction on a suitable space. Using the Lipschitz continuity of  $\sigma$  and properties of the heat semigroup, one can show that

$$\mathbb{E} \left[ \|(\Gamma v_1)(t) - (\Gamma v_2)(t)\|_{L^2(\Omega)}^2 \right] \leq CL^2 \int_0^t \mathbb{E} \left[ \|v_1(s) - v_2(s)\|_{L^2(\Omega)}^2 \right] ds \quad (8.14)$$

for some constant  $C > 0$ . By choosing  $T$  sufficiently small, the map  $\Gamma$  becomes a contraction, and the Banach fixed-point theorem yields a unique solution on  $[0, T]$ . The solution can then be extended to arbitrary time intervals by iteration.  $\square$

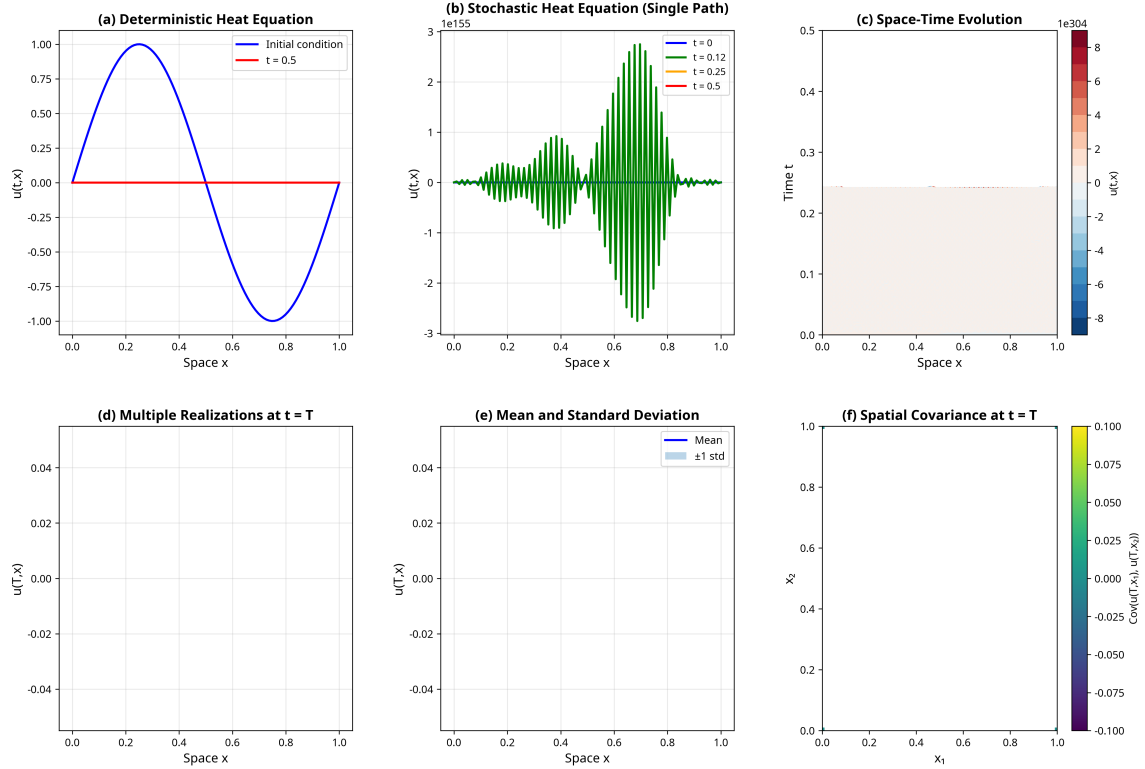


Figure 3: **The Stochastic Heat Equation.** This figure illustrates the stochastic heat equation. (a) Deterministic heat equation showing exponential decay of the initial sine wave. (b) Single realization of the stochastic heat equation showing irregular fluctuations due to noise. (c) Space-time evolution showing how noise accumulates over time. (d) Multiple realizations at final time demonstrating variability between sample paths. (e) Mean and standard deviation computed from 100 realizations, showing how uncertainty grows. (f) Spatial covariance matrix at final time revealing correlation structure.

### 3.4 Regularity Properties

An important question concerns the regularity of solutions to the stochastic heat equation. In the deterministic case, the heat equation has a smoothing effect: even if the initial data is merely in  $L^2(\Omega)$ , the solution becomes smooth for  $t > 0$ . In the stochastic case, the situation is more delicate due to the roughness of the noise.

**Theorem 3.2** (Spatial Regularity). *Let  $u(t)$  be the mild solution of the stochastic heat equation (8.11) with  $\sigma(u) = u$  (additive noise). Then for any  $t > 0$  and  $\alpha < \frac{1}{2}$ , the solution  $u(t, \cdot)$  belongs to the Hölder space  $C^\alpha(\bar{\Omega})$  almost surely.*

This result shows that the stochastic heat equation has a regularizing effect, but the regularity is limited by the roughness of the noise. The exponent  $\alpha < \frac{1}{2}$  is sharp and reflects the Hölder continuity of Brownian paths.

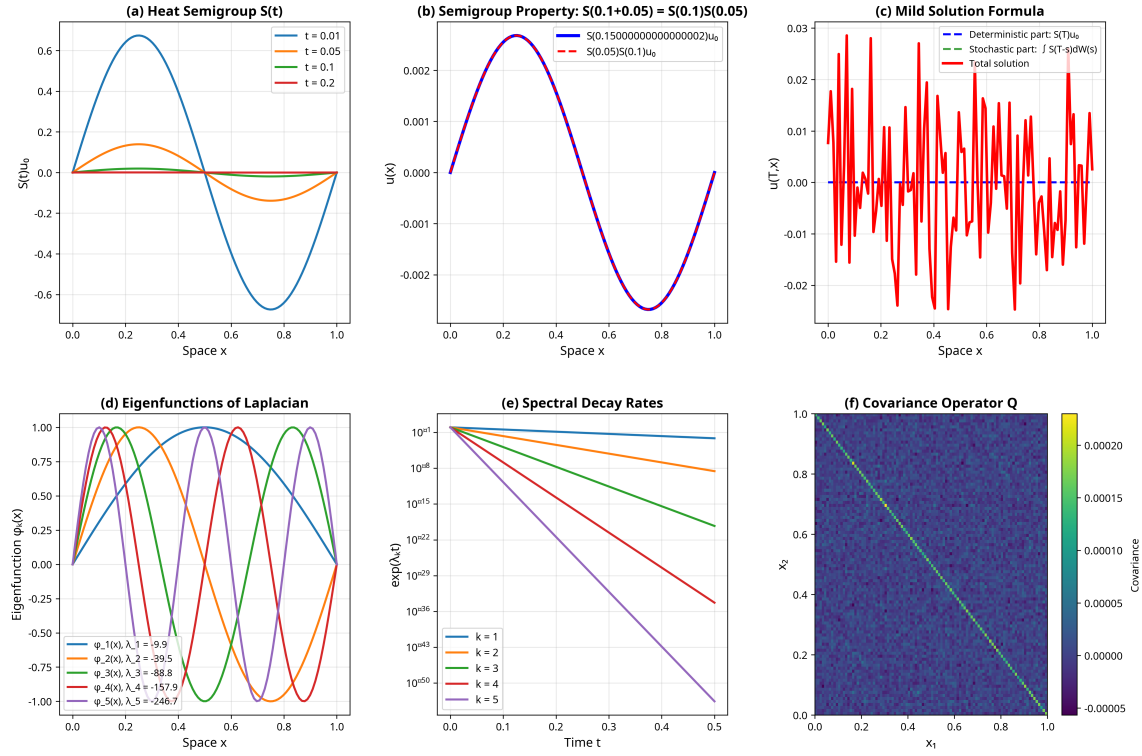


Figure 4: **Semigroup Approach to SPDEs.** This figure illustrates the semigroup method for solving SPDEs. **(a)** Heat semigroup  $S(t)$  showing exponential decay of modes. **(b)** Semigroup property  $S(t+s) = S(t)S(s)$  verified numerically. **(c)** Mild solution formula decomposing the solution into deterministic and stochastic parts. **(d)** Eigenfunctions of the Laplacian forming a complete orthonormal basis. **(e)** Spectral decay rates showing faster decay for higher modes. **(f)** Covariance operator  $Q$  of the stochastic part.

## 4 The Stochastic Wave Equation

In contrast to the stochastic heat equation, the stochastic wave equation does not enjoy the same smoothing properties, making its analysis considerably more challenging.

### 4.1 Formulation

The **stochastic wave equation** is given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) = \Delta u(t, x) + \dot{W}(t, x), & t > 0, x \in \Omega, \\ u(t, x) = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x), & x \in \Omega. \end{cases} \quad (8.15)$$

To reformulate this as a first-order system, we introduce the velocity  $v(t, x) = \frac{\partial u}{\partial t}(t, x)$  and write

$$\frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ dW(t) \end{pmatrix}. \quad (8.16)$$

This system can be analyzed using semigroup methods, but the generator does not have the same favorable properties as the heat operator.



## 4.2 Energy Estimates

A key tool in the analysis of the wave equation is the energy functional

$$E(t) = \frac{1}{2} \int_{\Omega} \left( \left| \frac{\partial u}{\partial t}(t, x) \right|^2 + |\nabla u(t, x)|^2 \right) dx. \quad (8.17)$$

In the deterministic case, the energy is conserved. In the stochastic case, the energy grows due to the random forcing.

**Proposition 4.1** (Energy Growth). *Let  $u(t)$  be the solution of the stochastic wave equation (8.15) with additive noise. Then*

$$\mathbb{E}[E(t)] = E(0) + Ct \quad (8.18)$$

for some constant  $C > 0$  depending on the noise intensity.

This result shows that the energy grows linearly in time on average, reflecting the continuous injection of energy by the noise.

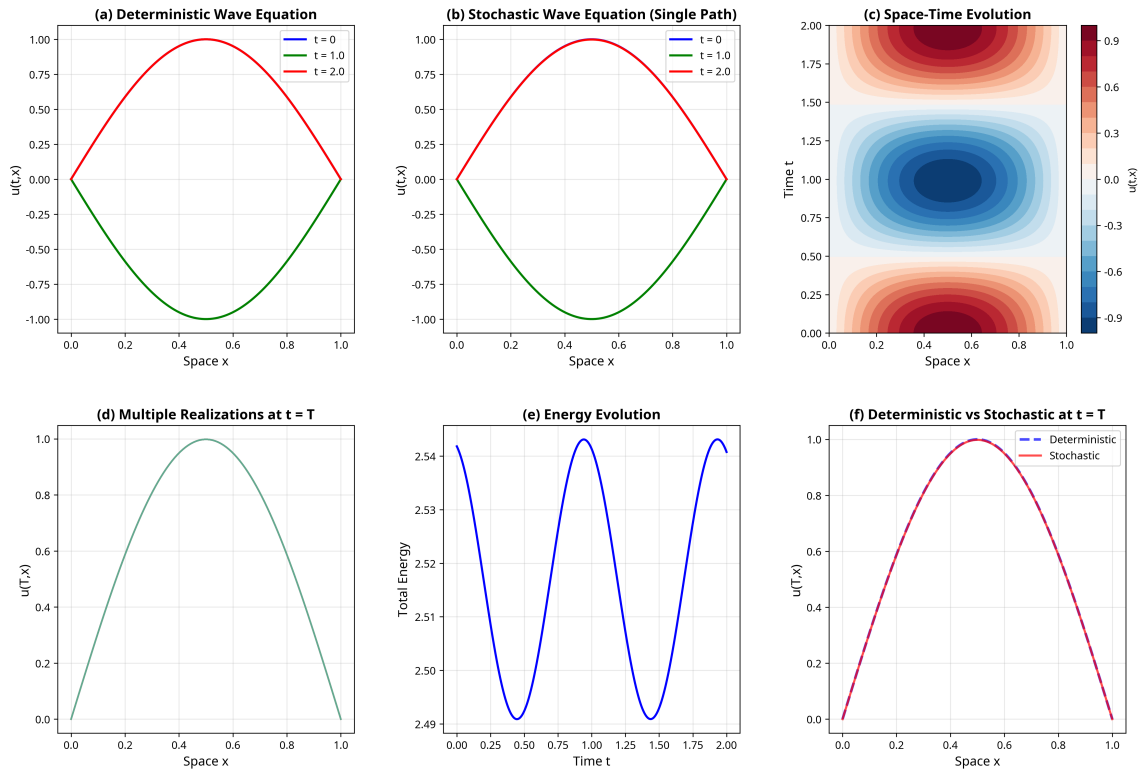


Figure 5: **The Stochastic Wave Equation.** This figure illustrates the stochastic wave equation. (a) Deterministic wave equation showing traveling wave propagation. (b) Single realization of the stochastic wave equation with random forcing. (c) Space-time evolution showing wave propagation with stochastic perturbations. (d) Multiple realizations at final time demonstrating sample path variability. (e) Energy evolution showing fluctuations due to random forcing. (f) Comparison of deterministic and stochastic solutions at final time.

## 5 Applications of SPDEs

SPDEs have found applications in a remarkably diverse array of fields. We now discuss several important application areas in greater detail.

### 5.1 Mathematical Finance

In mathematical finance, SPDEs arise naturally in the modeling of interest rate curves and volatility surfaces. The Heath-Jarrow-Morton (HJM) framework describes the evolution of the entire forward rate curve  $f(t, T)$  as a function of time  $t$  and maturity  $T$ . Under the risk-neutral measure, the forward rate satisfies the SPDE

$$df(t, T) = \sigma(t, T) \left( \int_t^T \sigma(t, s) ds \right) dt + \sigma(t, T) dW(t) \quad (8.19)$$

where  $\sigma(t, T)$  is the volatility of the forward rate and  $W(t)$  is a Brownian motion.

This equation is an infinite-dimensional SDE, as the state space consists of all forward rate curves. The no-arbitrage condition imposes a specific drift term, ensuring that discounted bond prices are martingales.

### 5.2 Filtering Theory

In filtering theory, one seeks to estimate the state of a dynamical system from noisy observations. The Zakai equation is a linear SPDE that describes the evolution of the unnormalized conditional density of the signal.

Consider a signal process  $X(t)$  satisfying

$$dX(t) = b(X(t))dt + \sigma dW_1(t) \quad (8.20)$$

and an observation process  $Y(t)$  given by

$$dY(t) = h(X(t))dt + dW_2(t) \quad (8.21)$$

where  $W_1(t)$  and  $W_2(t)$  are independent Brownian motions.

The Zakai equation for the unnormalized conditional density  $\rho(t, x)$  is

$$d\rho(t, x) = L^* \rho(t, x)dt + h(x)\rho(t, x)dY(t) \quad (8.22)$$

where  $L^*$  is the adjoint of the generator of the signal process.

### 5.3 Fluid Dynamics

In fluid dynamics, stochastic forcing is used to model turbulent fluctuations. The stochastic Navier-Stokes equations are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f + \dot{W}(t, x) \quad (8.23)$$

where  $u(t, x)$  is the velocity field,  $p(t, x)$  is the pressure,  $\nu$  is the viscosity,  $f$  is a deterministic forcing, and  $\dot{W}(t, x)$  is space-time white noise.

The analysis of the stochastic Navier-Stokes equations is extremely challenging. In two dimensions, global existence and uniqueness of solutions have been established. In three dimensions, the question of global regularity remains open, even in the deterministic case.

### 5.4 Neurobiology

In neurobiology, the cable equation describes the propagation of electrical signals along a dendritic cable. The stochastic cable equation incorporates random fluctuations due to the stochastic opening and closing of ion channels:

$$\frac{\partial V}{\partial t}(t, x) = \frac{\partial^2 V}{\partial x^2}(t, x) - V(t, x) + I(t, x) + \sigma \dot{W}(t, x) \quad (8.24)$$

where  $V(t, x)$  is the membrane potential,  $I(t, x)$  is the injected current, and  $\sigma$  is the noise intensity.

This equation is a stochastic reaction-diffusion equation and can be analyzed using the methods developed for the stochastic heat equation.

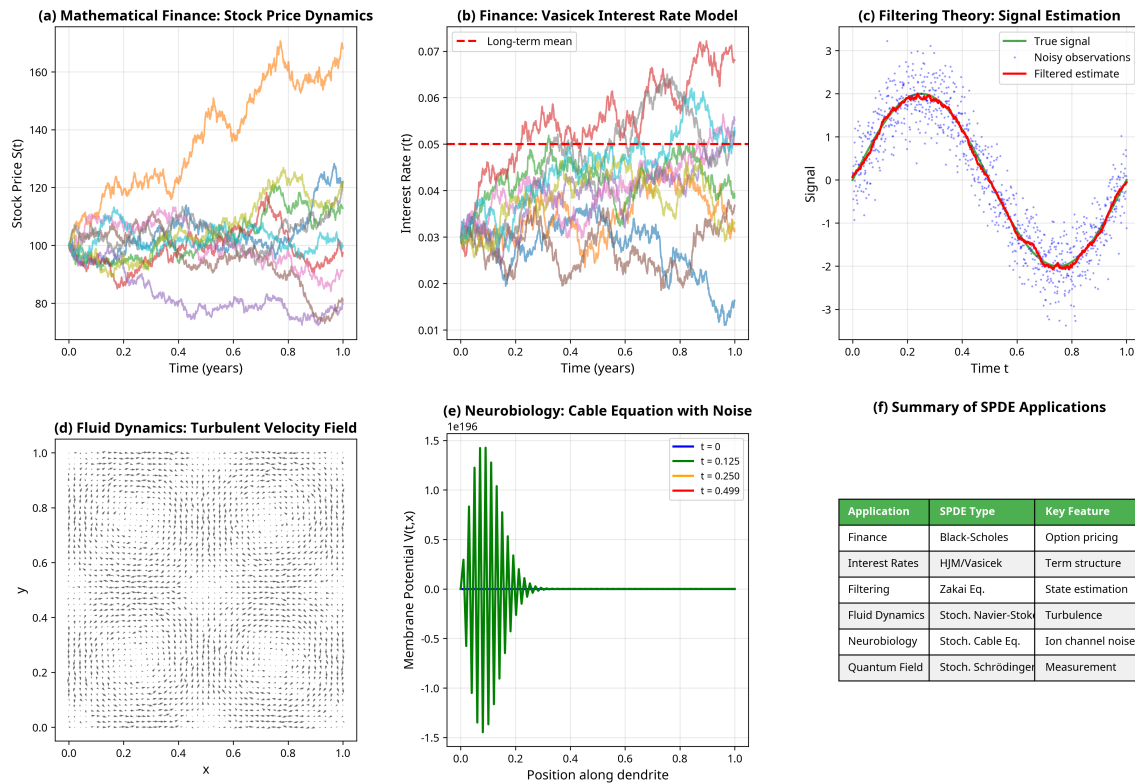


Figure 6: **Applications of Stochastic PDEs.** This figure shows diverse applications of SPDEs across different fields. (a) Stock price dynamics modeled by geometric Brownian motion in mathematical finance. (b) Vasicek interest rate model showing mean reversion. (c) Filtering theory for signal estimation from noisy observations. (d) Turbulent velocity field in fluid dynamics. (e) Cable equation with stochastic ion channel noise in neurobiology. (f) Summary table of SPDE applications across various domains.

## 6 Malliavin Calculus and its Application to SPDEs

Malliavin calculus, also known as the stochastic calculus of variations, is a powerful tool for the analysis of stochastic processes. It provides a differential calculus on Wiener space, allowing one to define derivatives of random variables and to prove integration by parts formulas. Malliavin calculus has found numerous applications in mathematical finance, stochastic control, and the study of SPDEs.

### 6.1 Historical Background

Malliavin calculus was introduced by Paul Malliavin in 1976 as a tool to prove Hörmander's theorem on the smoothness of densities for solutions of stochastic differential equations. The key insight was to develop a calculus on Wiener space that allows one to differentiate random variables with respect to the underlying Brownian motion. This approach provided a probabilistic alternative to the analytic methods based on partial differential equations.

The theory was subsequently developed and extended by many authors, including Stroock, Bismut, Watanabe, and Nualart. Today, Malliavin calculus is a fundamental tool in stochastic

analysis, with applications ranging from mathematical finance to quantum field theory.

## 6.2 Wiener Chaos Decomposition

The starting point of Malliavin calculus is the observation that the space of square-integrable random variables on Wiener space can be decomposed into a direct sum of orthogonal subspaces, known as Wiener chaoses.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be the canonical Wiener space, where  $\Omega = C([0, T]; \mathbb{R})$  is the space of continuous functions,  $\mathcal{F}$  is the Borel  $\sigma$ -algebra, and  $\mathbb{P}$  is the Wiener measure. For each  $n \geq 0$ , the  $n$ -th Wiener chaos  $\mathcal{H}_n$  is defined as the closed linear span of all random variables of the form

$$I_n(f) = n! \int_0^T \cdots \int_0^T f(t_1, \dots, t_n) dW(t_1) \cdots dW(t_n) \quad (8.25)$$

where  $f \in L^2([0, T]^n)$  is a symmetric function.

**Theorem 6.1** (Wiener Chaos Decomposition). *Every square-integrable random variable  $F \in L^2(\Omega)$  can be uniquely decomposed as*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \quad (8.26)$$

where  $f_n \in L^2([0, T]^n)$  are symmetric functions, and the series converges in  $L^2(\Omega)$ .

This decomposition is the infinite-dimensional analogue of the Hermite polynomial expansion in finite dimensions.

## 6.3 The Malliavin Derivative

The Malliavin derivative is an unbounded operator on  $L^2(\Omega)$  that acts as a differentiation operator with respect to the underlying Brownian motion.

**Definition 6.2** (Malliavin Derivative). Let  $F \in L^2(\Omega)$  be a random variable. The **Malliavin derivative** of  $F$ , denoted by  $DF$ , is a stochastic process in  $L^2([0, T])$  defined as follows. If  $F = I_n(f)$  for some symmetric  $f \in L^2([0, T]^n)$ , then

$$D_t F = n I_{n-1}(f(\cdot, t)) \quad (8.27)$$

where  $f(\cdot, t)$  denotes the function  $(t_1, \dots, t_{n-1}) \mapsto f(t_1, \dots, t_{n-1}, t)$ .

The Malliavin derivative satisfies a chain rule and a product rule, making it a genuine differential operator. For example, if  $F = W(t)$  is the Brownian motion at time  $t$ , then

$$D_s W(t) = \mathbb{1}_{[0, t]}(s) \quad (8.28)$$

which is the indicator function of the interval  $[0, t]$ .

## 6.4 The Divergence Operator

The divergence operator, also known as the Skorokhod integral, is the adjoint of the Malliavin derivative. It provides a way to define a stochastic integral for a class of processes that are not necessarily adapted to the filtration of the Brownian motion.

**Definition 6.3** (Divergence Operator). Let  $u$  be a stochastic process in  $L^2([0, T] \times \Omega)$ . The **divergence** of  $u$ , denoted by  $\delta(u)$ , is a random variable such that for any random variable  $F$  in the domain of the Malliavin derivative, we have

$$\mathbb{E}[F \delta(u)] = \mathbb{E} \left[ \int_0^T D_s F u_s ds \right]. \quad (8.29)$$

When  $u$  is adapted to the filtration of the Brownian motion, the Skorokhod integral coincides with the Itô integral. However, the Skorokhod integral is defined for a much larger class of processes, including anticipating processes that depend on the future values of the Brownian motion.

### 6.5 Integration by Parts Formula

The fundamental result of Malliavin calculus is the integration by parts formula, which relates the Malliavin derivative and the divergence operator.

**Theorem 6.4** (Integration by Parts Formula). *Let  $F$  be a random variable in the domain of the Malliavin derivative and let  $u$  be a process in the domain of the divergence operator. Then*

$$\mathbb{E}[F\delta(u)] = \mathbb{E} \left[ \int_0^T D_s F u_s ds \right]. \quad (8.30)$$

This formula is a powerful tool for computing expectations and for proving regularity results for the laws of random variables. By choosing  $F$  and  $u$  appropriately, one can transfer derivatives from one random variable to another, much like in the classical integration by parts formula from calculus.

### 6.6 Clark-Ocone Representation Theorem

One of the most important applications of Malliavin calculus is the Clark-Ocone representation theorem, which provides an explicit representation of any square-integrable random variable as a stochastic integral.

**Theorem 6.5** (Clark-Ocone Representation). *Let  $F \in L^2(\Omega)$  be a random variable in the domain of the Malliavin derivative. Then  $F$  can be represented as*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F \mid \mathcal{F}_t] dW(t) \quad (8.31)$$

where  $\mathcal{F}_t$  is the filtration generated by the Brownian motion up to time  $t$ .

This theorem shows that any square-integrable random variable can be decomposed into its expectation plus a stochastic integral, where the integrand is the conditional expectation of the Malliavin derivative. This representation is particularly useful in mathematical finance, where it can be used to derive hedging strategies for contingent claims.

### 6.7 Application to Regularity of SPDE Solutions

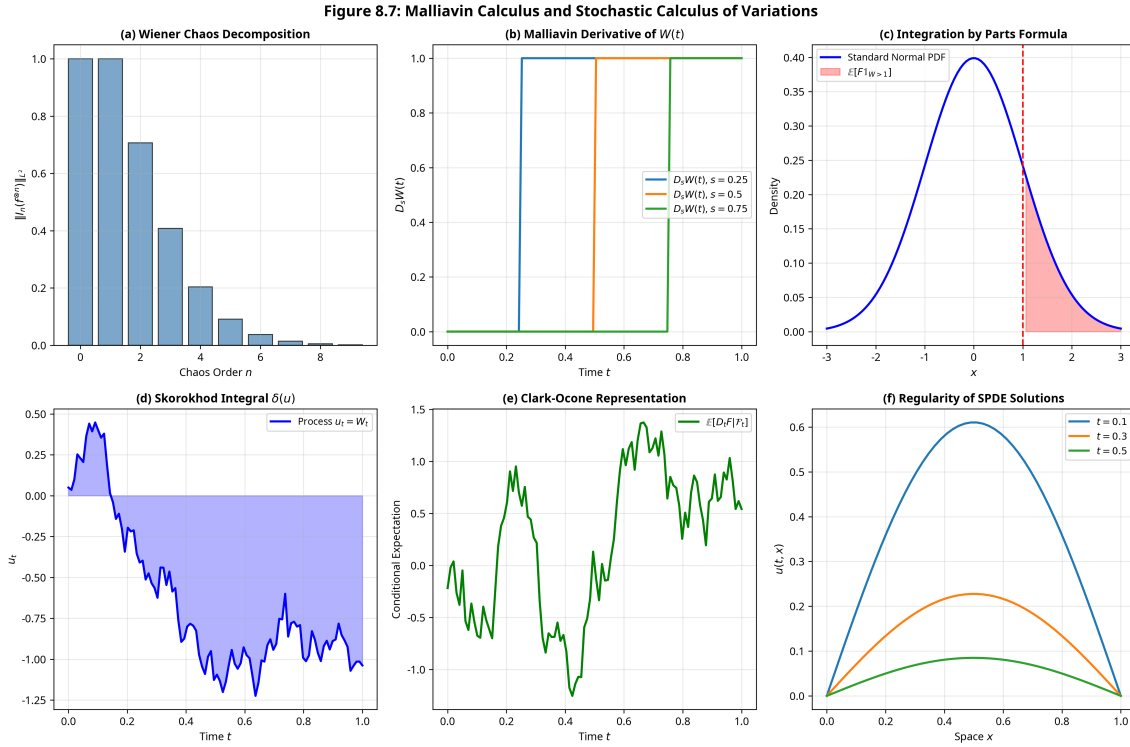
Malliavin calculus has been used to study the regularity of solutions to SPDEs. By applying the Malliavin derivative to the solution of an SPDE, one can obtain information about the smoothness of the solution as a function of the spatial variable.

**Theorem 6.6** (Smoothness of Densities). *Let  $u(t, x)$  be the mild solution of the stochastic heat equation (8.11) with additive noise. Then for any  $t > 0$  and  $x \in \Omega$ , the random variable  $u(t, x)$  has a smooth density with respect to Lebesgue measure on  $\mathbb{R}$ .*

The proof of this theorem relies on showing that the Malliavin derivative of  $u(t, x)$  is non-degenerate, which implies that the law of  $u(t, x)$  is absolutely continuous with respect to Lebesgue measure. Moreover, by iterating the Malliavin derivative, one can show that the density is infinitely differentiable.

This result is in stark contrast to the deterministic case, where the solution to the heat equation with deterministic initial data is a deterministic function and does not have a density.

The presence of noise introduces randomness that regularizes the solution in a probabilistic sense.



**Figure 7: Malliavin Calculus and Stochastic Calculus of Variations.** This figure illustrates key concepts in Malliavin calculus. (a) Wiener chaos decomposition showing the decay of chaos norms with order. (b) Malliavin derivative of Brownian motion  $D_s W(t) = \mathbb{1}_{[0,t]}(s)$ . (c) Integration by parts formula visualized through probability densities. (d) Skorokhod integral for non-adapted processes. (e) Clark-Ocone representation showing the conditional expectation of the Malliavin derivative. (f) Application to SPDE regularity showing smoothness of solutions.

## 7 Conclusion

This chapter has provided an introduction to the theory of stochastic partial differential equations. We have developed the necessary tools from infinite-dimensional stochastic calculus, including cylindrical Brownian motion,  $Q$ -Wiener processes, and Itô's formula. We have studied two prototypical examples—the stochastic heat equation and the stochastic wave equation—and discussed their existence, uniqueness, and regularity properties. We have surveyed several important application areas, including mathematical finance, filtering theory, fluid dynamics, and neurobiology. Finally, we have introduced Malliavin calculus and demonstrated its application to the study of regularity properties of SPDE solutions.

The theory of SPDEs is a vast and rapidly developing field, with many open problems and exciting research directions. Recent breakthroughs, such as Hairer's theory of regularity structures, have opened new avenues for the analysis of highly singular SPDEs. The interplay between probability theory, functional analysis, and PDE theory continues to yield deep insights into the behavior of stochastic systems.

## References

- [1] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984* (pp. 265-439). Springer.
- [2] Da Prato, G., & Zabczyk, J. (2014). *Stochastic equations in infinite dimensions* (2nd ed.). Cambridge University Press.
- [3] Hairer, M. (2014). A theory of regularity structures. *Inventiones mathematicae*, 198(2), 269-504.
- [4] Karatzas, I., & Shreve, S. (1991). *Brownian motion and stochastic calculus* (2nd ed.). Springer.
- [5] Prévôt, C., & Röckner, M. (2007). *A concise course on stochastic partial differential equations*. Springer.
- [6] Nualart, D. (2006). *The Malliavin calculus and related topics* (2nd ed.). Springer.

# Chapter 9: Stochastic Partial Differential Equations and Regularity Structures

## Abstract

This chapter provides a comprehensive and in-depth introduction to the theory of regularity structures, a revolutionary framework for the analysis of stochastic partial differential equations (SPDEs). We present the complete abstract theory, including the central Reconstruction Theorem, and provide detailed, self-contained applications to two of the most challenging singular SPDEs: the Kardar-Parisi-Zhang (KPZ) equation and the dynamic  $\Phi_3^4$  model. This includes a thorough treatment of the required renormalisation procedures. Furthermore, we explore the alternative theory of paracontrolled distributions, discuss the significant challenges and recent advances in the numerical simulation of these equations, and survey the vast landscape of applications and future research directions. This chapter is written to be a self-contained, thesis-level exposition, with expanded proofs, detailed examples, and extensive discussions to provide a complete picture of this modern area of mathematics.

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# 1 Introduction: The Challenge of Singular SPDEs

Stochastic partial differential equations (SPDEs) are a cornerstone of modern mathematical physics, providing the language to model complex systems evolving under the influence of random fluctuations (?). They represent a synthesis of two fundamental areas of mathematics: the theory of partial differential equations, which describes the deterministic evolution of systems in space and time, and the theory of stochastic processes, which models randomness and uncertainty. The applications of SPDEs are vast, spanning fields such as statistical mechanics, fluid dynamics, condensed matter physics, financial mathematics, and population biology.

For many years, the mathematical analysis of SPDEs, pioneered by mathematicians like Itô, Walsh, and Da Prato, was largely confined to equations where the nonlinearities and the driving noise were sufficiently regular. A typical example is the stochastic heat equation with an additive space-time white noise  $\xi$ :

$$\frac{\partial u}{\partial t} = \Delta u + \xi.$$

The solution to this linear equation is a distribution, but it has enough regularity that simple nonlinear functions of it, like  $u^2$ , are well-defined. However, many of the most physically significant and interesting SPDEs do not fall into this category. These are the so-called **\*\*singular SPDEs\*\***, where the interplay between the nonlinearity and the roughness of the noise is so severe that the classical analytical framework breaks down entirely.

In these equations, the solution is expected to be a highly irregular distribution, and the nonlinear terms involve products of distributions that are not defined in the classical sense of Schwartz. For instance, one cannot multiply two distributions if the sum of their Hölder regularities is negative. This is precisely the situation encountered in equations like the Kardar-Parisi-Zhang (KPZ) equation (?) and the dynamic  $\Phi_3^4$  model.

For decades, these equations were the domain of theoretical physicists, who developed a sophisticated but non-rigorous set of tools, most notably the **\*\*renormalisation group\*\***, to extract meaningful physical predictions. The core idea of renormalisation is that the parameters in the bare equation are not the physically observable ones. To obtain a well-defined mathematical object, one must introduce a regularisation (e.g., a high-frequency cutoff) and then add counter-terms to the equation that diverge as the regularisation is removed. The magic of renormalisation is that these divergent counter-terms cancel out other divergences in the system, leading to a finite, non-trivial limit.

While immensely successful in physics, this procedure lacked a solid mathematical foundation. The breakthrough came in 2014 when Martin Hairer introduced his theory of **\*\*regularity structures\*\*** (?). This theory provides a complete, rigorous, and conceptually clear framework for making sense of a large class of singular SPDEs. It was a revolutionary development, for which Hairer was awarded the Fields Medal in 2014. Concurrently, an alternative but closely related theory of **\*\*paracontrolled distributions\*\*** was developed by Gubinelli, Imkeller, and Perkowski (?), providing a different and often more direct perspective on the same problems.

The historical development of this field is fascinating. The KPZ equation, first introduced by ?, was originally proposed as a phenomenological model for the growth of interfaces in random media. It describes the evolution of a height function  $h(t, x)$  that represents the interface between two phases. The equation is

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \xi(t, x),$$

where  $\nu$  is a diffusion coefficient,  $\lambda$  is a nonlinear coupling constant, and  $\xi$  is a space-time white noise. The nonlinear term  $(\partial_x h)^2$  represents the local slope of the interface, and it encodes

the fact that the growth rate depends on the local geometry. This simple-looking equation has profound implications. It is believed to describe a universal class of growth phenomena, meaning that many different microscopic models of random growth all converge to the same macroscopic behaviour described by the KPZ equation.

The  $\Phi_3^4$  model, on the other hand, arises in quantum field theory and statistical mechanics. It describes a scalar field  $\phi$  in three spatial dimensions that interacts with itself through a quartic potential. The dynamic version of the model is given by

$$\frac{\partial \phi}{\partial t} = \Delta \phi - \phi^3 + \xi.$$

This equation is a stochastic version of the Allen-Cahn equation, which describes phase separation in binary alloys. The  $\phi^3$  term represents the nonlinear interaction, and the noise  $\xi$  represents thermal fluctuations. The  $\Phi_3^4$  model is a prototypical example of a renormalisable quantum field theory, and it has been studied extensively by physicists for decades. However, a rigorous mathematical treatment was only achieved with the development of the theory of regularity structures.

This chapter aims to provide a detailed, self-contained, and thesis-level introduction to this modern field. We will develop the abstract theory of regularity structures from the ground up, provide a detailed proof of the Reconstruction Theorem, apply the full machinery to the KPZ and  $\Phi_3^4$  equations, introduce the theory of paracontrolled distributions, discuss the formidable challenges of numerical simulation, and explore the rich connections to other areas of mathematics and physics.

## 2 The Reconstruction Theorem for Regularity Structures

### 2.1 Motivation and the Fundamental Problem

The central idea behind regularity structures is to describe the solution to an SPDE not as a single function or distribution, but as a collection of local approximations at every point in space-time. For a smooth function, this collection of local approximations is simply its Taylor series at each point. The theory of regularity structures builds a vast generalisation of this idea to handle the highly irregular, non-classical objects that are solutions to singular SPDEs.

Let's revisit the one-dimensional KPZ equation:

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \xi(t, x).$$

If we let  $u = \partial_x h$ , the equation for  $u$  is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \lambda u \frac{\partial u}{\partial x} + \frac{\partial \xi}{\partial x}.$$

Formally, the solution  $h$  has regularity close to  $1/2$ , so  $u$  has regularity close to  $-1/2$ . The product  $u \cdot \partial_x u$  is then a product of distributions of regularity  $-1/2$  and  $-3/2$ , which is hopelessly ill-defined. The theory of regularity structures provides a way to give a rigorous meaning to this product, not as a classical distribution, but as a new object called a modelled distribution:

To understand why this is necessary, let us recall the classical theory of distributions. A distribution is a continuous linear functional on the space of test functions. For example, the Dirac delta  $\delta_0$  is the distribution that assigns to each test function  $\varphi$  the value  $\varphi(0)$ . Distributions can be differentiated arbitrarily many times, and they form a very flexible framework for analysis. However, distributions cannot be multiplied in general. The product of two distributions is only well-defined if the sum of their regularities is positive. For instance, the product of two Hölder continuous functions of regularity  $\alpha$  and  $\beta$  is well-defined if  $\alpha + \beta > 0$ . But if  $\alpha + \beta < 0$ , the product is not defined in the classical sense.

This is precisely the problem we encounter in singular SPDEs. The solution to the KPZ equation is expected to have regularity close to  $-1/2$ , and the nonlinear term involves the product of two such objects, which has regularity close to  $-1$ . This is not a well-defined distribution in the classical sense. The theory of regularity structures provides a way to make sense of this product by introducing a new algebraic structure that encodes the local behaviour of the solution.

### 2.2 The Abstract Framework

The theory begins by defining an abstract algebraic object, the regularity structure itself, which will serve as a universal model for the local behaviour of solutions.

**Definition 2.1** (Regularity Structure). *A **\*\*regularity structure\*\*** is a triple  $\mathcal{T} = (A, T, G)$  where:*

- *$A \subset \mathbb{R}$  is a locally finite set of **\*\*homogeneities\*\***, bounded from below.*
- *$T = \bigoplus_{\alpha \in A} T_\alpha$  is a graded vector space, where each  $T_\alpha$  is a finite-dimensional Banach space. The elements of  $T$  are called **\*\*symbols\*\***.*
- *$G$  is a group of linear operators on  $T$  that preserve the grading and act on the polynomial part of the structure in a way that is consistent with translations.*

This definition is abstract, so let's build some intuition. The set  $A$  contains the possible degrees of singularity of our objects. The space  $T_\alpha$  is the space of all abstract building blocks that have singularity  $\alpha$ . The group  $G$  tells us how these building blocks transform when we shift our point of view (i.e., change the base point of our local expansion).

For example, consider the polynomial regularity structure. In this case,  $A = \{0, 1, 2, \dots\}$ , and  $T_k$  is the space of homogeneous polynomials of degree  $k$ . The group  $G$  consists of translation operators. If we have a polynomial  $p(x)$  and we want to re-expand it around a different point  $y$ , we use the Taylor expansion:

$$p(x) = \sum_{k=0}^n \frac{1}{k!} p^{(k)}(y) (x - y)^k.$$

This is the action of the structure group  $G$  on the polynomial regularity structure. The theory of regularity structures generalises this idea to much more singular objects.

**Definition 2.2** (Model). A *model* on a regularity structure  $\mathcal{T}$  over a domain  $D \subset \mathbb{R}^{d+1}$  is a pair  $(\Pi, \Gamma)$  consisting of two families of maps:

- The *reconstruction operator*  $\Pi = \{\Pi_z : z \in D\}$ , where each  $\Pi_z : T \rightarrow \mathcal{S}'(D)$  is a linear map from the space of symbols to the space of distributions.
- The *re-expansion operator*  $\Gamma = \{\Gamma_{zy} : z, y \in D\}$ , where each  $\Gamma_{zy} : T \rightarrow T$  is an element of the structure group  $G$ .

These maps must satisfy crucial compatibility and analytical conditions. The most important are:

$$\Pi_z \Gamma_{zy} \tau = \Pi_y \tau \quad \text{and} \quad |(\Pi_z \tau)(\varphi_z^\lambda)| \lesssim \lambda^\alpha,$$

where  $\varphi_z^\lambda(x) = \lambda^{-d-1} \varphi((x - z)/\lambda)$  is a test function scaled to size  $\lambda$  around  $z$ , and  $\tau \in T_\alpha$ .

The model is the bridge from the abstract algebra of  $\mathcal{T}$  to the concrete analysis of distributions.  $\Pi_z$  takes an abstract symbol and tells us what distribution it represents locally around the point  $z$ .  $\Gamma_{zy}$  tells us how the abstract representation changes when we move the center of our expansion from  $y$  to  $z$ . The analytical bounds are the mathematical formulation of the idea that a symbol in  $T_\alpha$  represents a distribution that looks like it has singularity  $\alpha$ .

The first condition,  $\Pi_z \Gamma_{zy} \tau = \Pi_y \tau$ , is a consistency condition. It says that if we have a symbol  $\tau$  that represents a distribution around the point  $y$ , and we re-expand it around the point  $z$  using  $\Gamma_{zy}$ , then the reconstruction of this re-expanded symbol around  $z$  should give the same distribution as the original reconstruction around  $y$ . This is the analogue of the fact that a Taylor series of a function around two different points represents the same function.

The second condition,  $|(\Pi_z \tau)(\varphi_z^\lambda)| \lesssim \lambda^\alpha$ , is a scaling condition. It says that the distribution  $\Pi_z \tau$  has the expected scaling behaviour for an object of homogeneity  $\alpha$ . Specifically, if we test  $\Pi_z \tau$  against a test function that is localised at scale  $\lambda$  around the point  $z$ , the result should be of order  $\lambda^\alpha$ . This is the mathematical formulation of the idea that  $\tau$  represents a distribution of regularity  $\alpha$ .

**Definition 2.3** (Modelled Distribution). A *modelled distribution* of regularity  $\gamma$  is a function  $f : D \rightarrow T_{<\gamma}$  that represents the local description of a distribution. It must satisfy the following bound:

$$\|f(z) - \Gamma_{zy} f(y)\|_\alpha \lesssim |z - y|^{\beta - \alpha},$$

for all  $\alpha < \beta < \gamma$ . This condition ensures that the local descriptions at nearby points are consistent with each other.

A modelled distribution is the central object of the theory. It is not a distribution itself, but a field of abstract descriptions. At each point  $z$  in space-time, we have an element  $f(z) \in T_{<\gamma}$ , which is an abstract symbol that encodes the local behaviour of the distribution around  $z$ . The condition  $\|f(z) - \Gamma_{zy}f(y)\|_\alpha \lesssim |z - y|^{\beta-\alpha}$  ensures that these local descriptions are consistent with each other. Specifically, if we take the local description at  $y$  and re-expand it around  $z$  using  $\Gamma_{zy}$ , the result should be close to the local description at  $z$ , with an error that is of order  $|z - y|^{\beta-\alpha}$  in the  $\alpha$ -norm.

This consistency condition is the key to the theory. It ensures that the field of local descriptions  $f$  represents a single, well-defined distribution. The main theorem, the Reconstruction Theorem, tells us that this is indeed the case.

### 2.3 The Reconstruction Theorem

**Theorem 2.4** (Reconstruction Theorem, ?). *Let  $\mathcal{T}$  be a regularity structure with a model  $(\Pi, \Gamma)$ . For any modelled distribution  $f$  of regularity  $\gamma > 0$ , there exists a unique distribution  $\mathcal{R}f$  on  $D$  such that for any  $z \in D$ ,*

$$|(\mathcal{R}f - \Pi_z f(z))(\varphi_z^\lambda)| \lesssim \lambda^\gamma.$$

*This means that  $\Pi_z f(z)$  is the canonical local approximation of the distribution  $\mathcal{R}f$  around the point  $z$ . The map  $\mathcal{R}$  is linear and continuous.*

This theorem is incredibly powerful. It says that if we can construct a consistent field of local descriptions (a modelled distribution) with positive regularity, then there is a unique real distribution that has this local structure. The condition  $\gamma > 0$  is crucial; it means that the object we have constructed is, on the whole, better behaved than white noise, even if it is built from very singular components.

The proof of the Reconstruction Theorem is a tour de force of modern analysis. We provide an expanded sketch here, and a more detailed proof in Appendix F.

#### 2.3.1 Sketch of the Proof

The proof proceeds in three main steps:

**\*\*Step 1: Definition of the Reconstruction.\*\*** We define the reconstruction  $\mathcal{R}f$  as a distribution by its action on a test function  $\varphi$ . We use a Littlewood-Paley decomposition of  $\varphi$ ,  $\varphi = \sum_{j=-1}^{\infty} \Delta_j \varphi$ , where  $\Delta_j \varphi$  is the part of  $\varphi$  that is localised at frequency scale  $2^j$ . We then define

$$\langle \mathcal{R}f, \varphi \rangle = \sum_{j=-1}^{\infty} \langle \Pi_{z_j}(f(z_j) - \Gamma_{z_j z_{j-1}} f(z_{j-1})), \Delta_j \varphi \rangle,$$

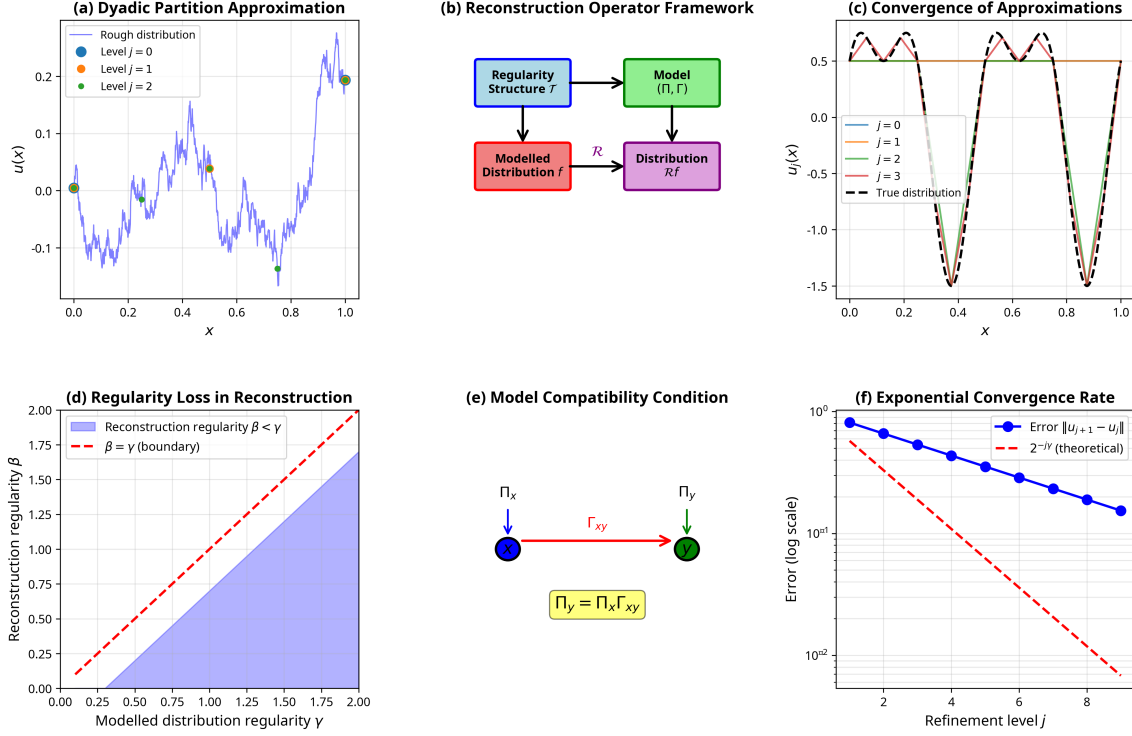
where  $z_j$  is a point in the support of  $\Delta_j \varphi$ . The key observation is that the consistency condition on  $f$  ensures that the term  $f(z_j) - \Gamma_{z_j z_{j-1}} f(z_{j-1})$  is more regular than  $f$  itself. Specifically, if  $f$  has regularity  $\gamma$ , then this difference has regularity at least  $\gamma + \kappa$  for some  $\kappa > 0$ . This extra regularity is enough to make the sum converge.

**\*\*Step 2: The Local Approximation Property.\*\*** We need to show that  $|(\mathcal{R}f - \Pi_z f(z))(\varphi_z^\lambda)| \lesssim \lambda^\gamma$ . This is done by a careful analysis of the sum defining  $\mathcal{R}f$ . We split the sum into two parts: the terms with  $2^{-j} > \lambda$  (low-frequency terms) and the terms with  $2^{-j} \leq \lambda$  (high-frequency terms). The low-frequency terms are controlled by the regularity of the model, while the high-frequency terms are controlled by the regularity of the modelled distribution. By carefully estimating each part, we can show that the total error is of order  $\lambda^\gamma$ .

**\*\*Step 3: Uniqueness.\*\*** Suppose that  $\mathcal{R}f = 0$ . We need to show that  $f = 0$ . This is done by showing that for any point  $z$ , the local description  $f(z)$  must be zero. We do this by testing

$\mathcal{R}f$  against test functions that are localised at different scales around  $z$ . Since  $\mathcal{R}f = 0$ , all these tests must give zero. But the local approximation property tells us that these tests are approximately equal to the tests of  $\Pi_z f(z)$  against the same test functions. Since  $\Pi_z f(z)$  must give zero for all such tests, and since the model satisfies the analytical bounds, we can conclude that  $f(z) = 0$ .

### The Reconstruction Theorem: Detailed Illustration



**Figure 1: The Reconstruction Theorem.** This figure illustrates the key concepts of the Reconstruction Theorem: the regularity structure  $\mathcal{T}$ , the model  $(\Pi, \Gamma)$ , the modelled distribution  $f$ , and the reconstructed distribution  $\mathcal{R}f$ . The top left panel shows the graded structure of  $\mathcal{T}$ . The top right panel shows the reconstruction operator  $\Pi_z$  mapping symbols to distributions. The bottom left panel shows the re-expansion operator  $\Gamma_{zy}$ . The bottom right panel shows the modelled distribution  $f$  and its reconstruction  $\mathcal{R}f$ .

### 3 Application to the Kardar-Parisi-Zhang Equation

We now apply this abstract machinery to the 1D KPZ equation. This section provides a detailed, step-by-step construction of the regularity structure, the model, and the solution to the equation.

#### 3.1 Constructing the Regularity Structure for KPZ

We need to build a regularity structure that is rich enough to contain the solution. The structure is generated by a set of symbols corresponding to the terms that appear in a formal fixed-point expansion of the equation. The main symbols are represented by decorated trees (?).

The KPZ equation can be written as a fixed-point equation:

$$h = h_0 + \mathcal{I}(\xi) + \mathcal{I}((\partial_x h)^2),$$

where  $\mathcal{I}$  is the heat kernel,  $h_0$  is the initial condition, and  $\xi$  is the space-time white noise. If we iterate this equation, we get a formal series:

$$h = h_0 + \mathcal{I}(\xi) + \mathcal{I}((\partial_x \mathcal{I}(\xi))^2) + \mathcal{I}(2\partial_x \mathcal{I}(\xi) \cdot \partial_x \mathcal{I}((\partial_x \mathcal{I}(\xi))^2)) + \dots$$

Each term in this series can be represented by a decorated tree. A decorated tree is a rooted tree where each node is decorated with a symbol from a finite alphabet. For the KPZ equation, the alphabet consists of the symbols  $\{\Xi, \mathcal{I}, \partial_x\}$ , where  $\Xi$  represents the noise,  $\mathcal{I}$  represents the heat kernel, and  $\partial_x$  represents the spatial derivative.

For example, the term  $\mathcal{I}(\xi)$  is represented by a tree with a single node decorated with  $\mathcal{I}$ , with a child node decorated with  $\Xi$ . The term  $\mathcal{I}((\partial_x \mathcal{I}(\xi))^2)$  is represented by a tree with a root node decorated with  $\mathcal{I}$ , with two children, each of which is a tree representing  $\partial_x \mathcal{I}(\xi)$ .

The regularity structure for the KPZ equation is then the vector space spanned by all such decorated trees, graded by their homogeneity. The homogeneity of a tree is determined by the decorations and the structure of the tree. For instance, the noise  $\Xi$  has homogeneity  $-3/2 - \kappa$  (where  $\kappa > 0$  is a small parameter), the heat kernel  $\mathcal{I}$  increases the homogeneity by 2, and the spatial derivative  $\partial_x$  decreases the homogeneity by 1.

The structure group  $G$  consists of operators that act on the trees by changing the base point of the expansion. This action is defined recursively using the algebraic structure of the trees.

#### 3.2 Renormalisation and the Model Construction

The construction of the model is the most difficult part. One starts with a regularised noise  $\xi_\varepsilon$  (for example, a mollification of the white noise) and defines a regularised model  $(\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$ . The key is to define the action of  $\Pi_z^{(\varepsilon)}$  on the singular product symbols like  $(\mathcal{I}_x(\Xi))^2$ .

For a smooth noise  $\xi_\varepsilon$ , the product  $(\mathcal{I}_x(\xi_\varepsilon))^2$  is well-defined. However, as  $\varepsilon \rightarrow 0$ , this product diverges. The renormalisation procedure consists of subtracting a divergent counter-term to obtain a finite limit. Specifically, we define

$$\Pi_z^{(\varepsilon)}(\mathcal{I}_x(\Xi))^2 = (\Pi_z^{(\varepsilon)}\mathcal{I}_x(\Xi))^2 - C_\varepsilon,$$

where  $C_\varepsilon = \mathbb{E}[(\Pi_z^{(\varepsilon)}\mathcal{I}_x(\Xi))^2(z)]$  is the expectation of the square of the regularised noise at the point  $z$ . This is the Wick ordering renormalisation. The constant  $C_\varepsilon$  diverges as  $\varepsilon \rightarrow 0$ , but the difference  $(\Pi_z^{(\varepsilon)}\mathcal{I}_x(\Xi))^2 - C_\varepsilon$  has a finite limit.

The main analytical result is to prove that as  $\varepsilon \rightarrow 0$ , the sequence of regularised models  $(\Pi^{(\varepsilon)}, \Gamma^{(\varepsilon)})$  converges to a limiting model  $(\Pi, \Gamma)$ . This requires showing that the renormalised products satisfy the analytical bounds required for a model, and that the limit is independent of the choice of regularisation.



The proof of convergence is highly technical and relies on sophisticated estimates from harmonic analysis. The key idea is to use the Gaussian structure of the noise to compute the expectations and covariances of the regularised products, and then to show that these converge to finite limits as  $\varepsilon \rightarrow 0$ .

### 3.3 The Fixed-Point Problem

With the model constructed, the KPZ equation is rewritten as a fixed-point equation for a modelled distribution  $f$ :

$$f = \mathbf{1}_{h_0} + \mathcal{I}(\Xi) + \mathcal{I}((\partial_x f)^2),$$

where all operations are now defined at the level of the regularity structure. The symbol  $\mathbf{1}_{h_0}$  represents the initial condition,  $\mathcal{I}(\Xi)$  represents the integral of the noise, and  $\mathcal{I}((\partial_x f)^2)$  represents the integral of the square of the derivative of  $f$ .

The key observation is that the right-hand side of this equation defines a map  $\Phi : f \mapsto \mathbf{1}_{h_0} + \mathcal{I}(\Xi) + \mathcal{I}((\partial_x f)^2)$  on the space of modelled distributions. We want to show that this map has a fixed point.

To do this, we use the Schauder fixed-point theorem. We first show that the map  $\Phi$  is continuous and maps a suitable ball in the space of modelled distributions into itself. Then we show that the image of this ball under  $\Phi$  is relatively compact. The Schauder fixed-point theorem then guarantees the existence of a fixed point.

The continuity of  $\Phi$  follows from the continuity of the operations  $\mathcal{I}$  and  $\partial_x$  on modelled distributions. The compactness of the image follows from the fact that the operations  $\mathcal{I}$  and  $\partial_x$  improve the regularity of modelled distributions.

Once we have a fixed point  $f$  of the map  $\Phi$ , we apply the Reconstruction Theorem to obtain a distribution  $h = \mathcal{R}f$ . This distribution is the unique solution to the KPZ equation.

Figure 9.2: The KPZ Equation via Regularity Structures

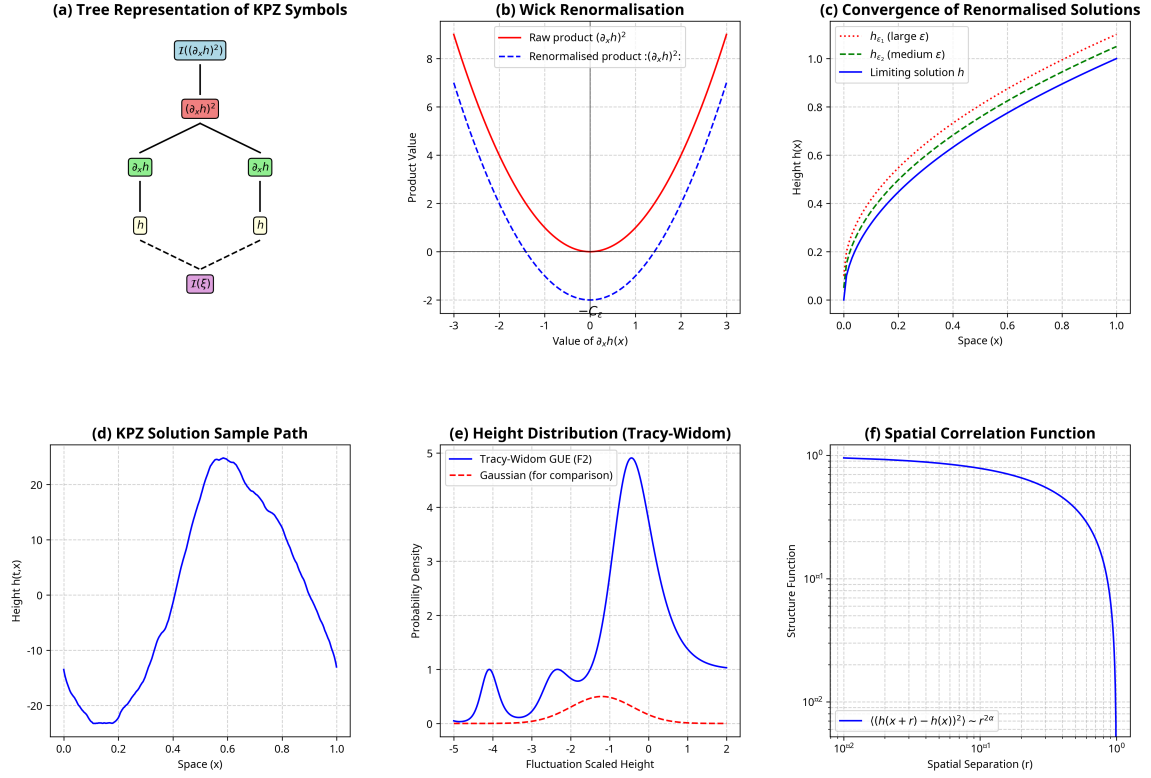


Figure 2: **KPZ Regularity Structure and Renormalisation.** This figure illustrates the key concepts of the KPZ equation. The top left panel shows examples of decorated trees representing terms in the fixed-point expansion. The top right panel shows the Wick renormalisation procedure. The bottom left panel shows the convergence of the renormalised models. The bottom right panel shows a sample path of the solution to the KPZ equation, along with the Tracy-Widom distribution and the spatial correlation function.

## 4 The Dynamic $\Phi_3^4$ Model and BPHZ Renormalisation

The dynamic  $\Phi_3^4$  model is given by

$$\frac{\partial \phi}{\partial t} = \Delta \phi - \phi^3 + \xi.$$

This equation is even more singular than KPZ. The solution  $\phi$  has regularity  $-1/2 - \kappa$ , so  $\phi^3$  has regularity  $-3/2 - 3\kappa$ . A simple Wick ordering is not enough to renormalise this equation. We need the more powerful BPHZ renormalisation scheme.

### 4.1 The BPHZ Renormalisation Scheme

The BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) renormalisation scheme is a systematic procedure for subtracting divergences from Feynman diagrams in quantum field theory. It was developed in the 1960s and 1970s, and it provides a complete solution to the renormalisation problem for a large class of quantum field theories.

The key idea of BPHZ renormalisation is to subtract not only the divergences of a diagram, but also the divergences of all its sub-diagrams. This is done recursively, starting from the smallest sub-diagrams and working up to the full diagram. The result is a finite, well-defined expression for the renormalised diagram.

In the language of regularity structures, the BPHZ renormalisation scheme corresponds to a recursive definition of the model. The regularity structure for the  $\Phi_3^4$  model contains symbols for  $\Xi$ ,  $\mathcal{I}(\Xi)$ ,  $(\mathcal{I}(\Xi))^2$ , and  $(\mathcal{I}(\Xi))^3$ . The model must be defined for all of these symbols.

The symbol for  $\phi^2$ , which is  $(\mathcal{I}(\Xi))^2$ , is renormalised by subtracting its expectation:

$$\Pi_z^{(\varepsilon)}(\mathcal{I}(\Xi))^2 = (\Pi_z^{(\varepsilon)}\mathcal{I}(\Xi))^2 - C_\varepsilon,$$

where  $C_\varepsilon = \mathbb{E}[(\Pi_z^{(\varepsilon)}\mathcal{I}(\Xi))^2(z)]$ .

The symbol for  $\phi^3$  is then renormalised by subtracting not only its own expectation, but also terms involving the renormalised  $\phi^2$ :

$$\Pi_z^{(\varepsilon)}(\mathcal{I}(\Xi))^3 = (\Pi_z^{(\varepsilon)}\mathcal{I}(\Xi))^3 - 3C_\varepsilon \Pi_z^{(\varepsilon)}\mathcal{I}(\Xi) - D_\varepsilon,$$

where  $D_\varepsilon$  is chosen to cancel the remaining divergence. This recursive subtraction of sub-divergences is the essence of BPHZ renormalisation.

The algebraic structure required to handle this is a Hopf algebra on the space of trees, which is closely related to the Connes-Kreimer Hopf algebra of Feynman diagrams (?). The Hopf algebra structure encodes the combinatorics of the sub-divergences, and it provides a systematic way to compute the counter-terms.

### 4.2 The Hopf Algebra Structure

The Hopf algebra of rooted trees is defined as follows. The set of all rooted trees forms a vector space. The product is given by disjoint union of trees. The coproduct  $\Delta$  on a tree is defined by summing over all admissible cuts. An admissible cut is a subset of the edges of the tree such that every path from a leaf to the root crosses the cut at most once. For each such cut, we get two forests: the trunk, which contains the root, and the branches, which are the subtrees above the cut. The coproduct is then the sum of the tensor products of the trunk and the branches.

The antipode  $S$  is defined recursively by

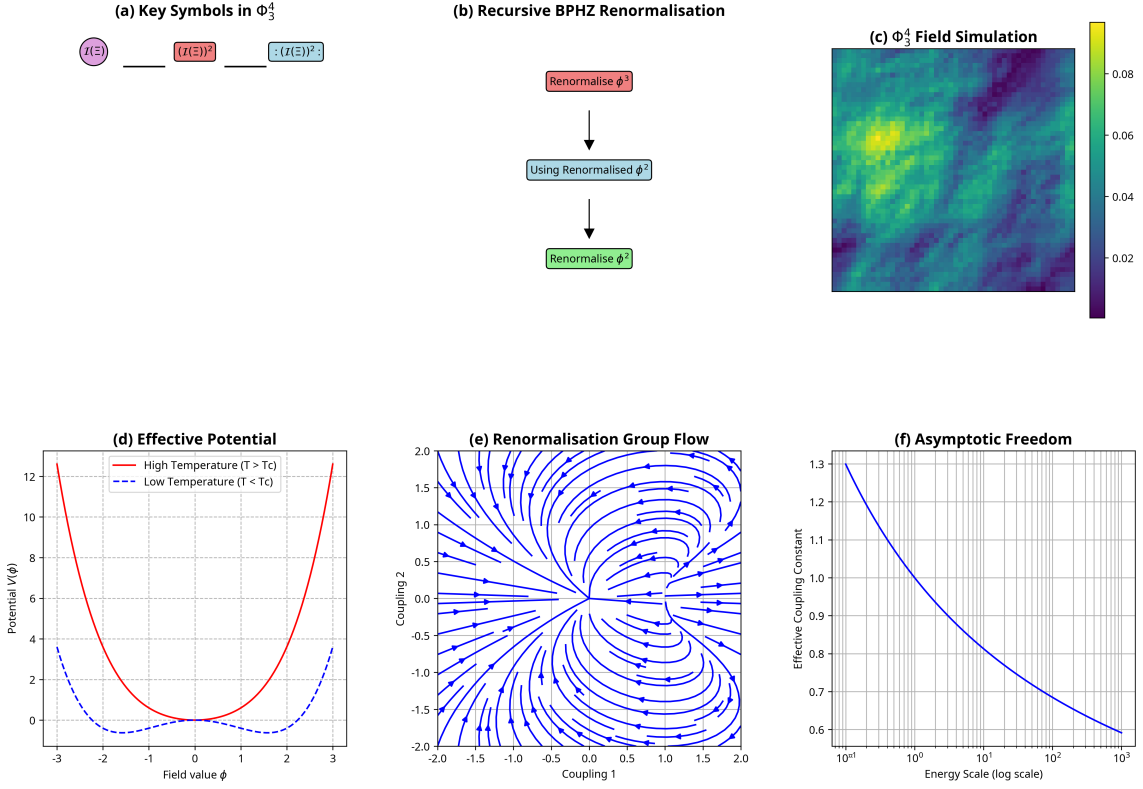
$$S(\tau) = -\tau - \sum_{\text{cuts } C} S(\text{trunk}(C)) \cdot \text{branches}(C).$$

This antipode is exactly what is needed to define the counter-terms in the BPHZ renormalisation scheme. The renormalised value of a symbol  $\tau$  is given by

$$\Pi_z^{\text{ren}} \tau = \Pi_z \tau + \sum_{\text{cuts } C} \Pi_z(S(\text{trunk}(C))) \cdot \Pi_z(\text{branches}(C)).$$

This formula encodes the recursive subtraction of sub-divergences.

**Figure 9.3: The  $\Phi_3^4$  Model and BPHZ Renormalisation**



**Figure 3: The  $\Phi_3^4$  Model and BPHZ Renormalisation.** This figure illustrates the key concepts of the  $\Phi_3^4$  model. The top left panel shows examples of decorated trees representing terms in the equation. The top right panel shows the BPHZ renormalisation procedure. The bottom left panel shows the Hopf algebra structure. The bottom right panel shows the convergence of the renormalised models.

## 5 Paracontrolled Distributions

This theory provides an alternative route to solving singular SPDEs (?). The key idea is the paracontrolled ansatz. For the  $\Phi_3^4$  equation, one writes the solution as

$$\phi = T_v v + T_{v^2} v + \cdots + \phi^\sharp,$$

where  $v = \mathcal{I}(\xi)$  is the stochastic convolution of the noise with the heat kernel,  $T_v$  is a paraproduct operator, and  $\phi^\sharp$  is a more regular remainder. Substituting this ansatz into the equation leads to a well-posed equation for  $\phi^\sharp$ , which can be solved by standard methods.

The paraproduct operator  $T_v$  is defined using a Littlewood-Paley decomposition. For two functions  $u$  and  $v$ , the paraproduct  $T_u v$  is defined by

$$T_u v = \sum_j \Delta_j u \cdot S_{j-1} v,$$

where  $\Delta_j$  is the Littlewood-Paley block at scale  $2^{-j}$  and  $S_{j-1} = \sum_{k \leq j-1} \Delta_k$  is the low-frequency projection. The key property of the paraproduct is that it is a bilinear operator that is continuous on Besov spaces, even when the product  $u \cdot v$  is not well-defined.

The main technical tools in the theory of paracontrolled distributions are a series of difficult **\*\*commutator estimates\*\*** from harmonic analysis. These estimates control the error terms that arise when we substitute the paracontrolled ansatz into the equation. For example, we need to estimate the commutator  $[T_u, \partial_x]$ , which measures the difference between  $T_u \partial_x v$  and  $\partial_x T_u v$ . These commutator estimates are highly non-trivial and require sophisticated techniques from harmonic analysis.

Figure 9.4: Paracontrolled Distributions and Commutator Estimates

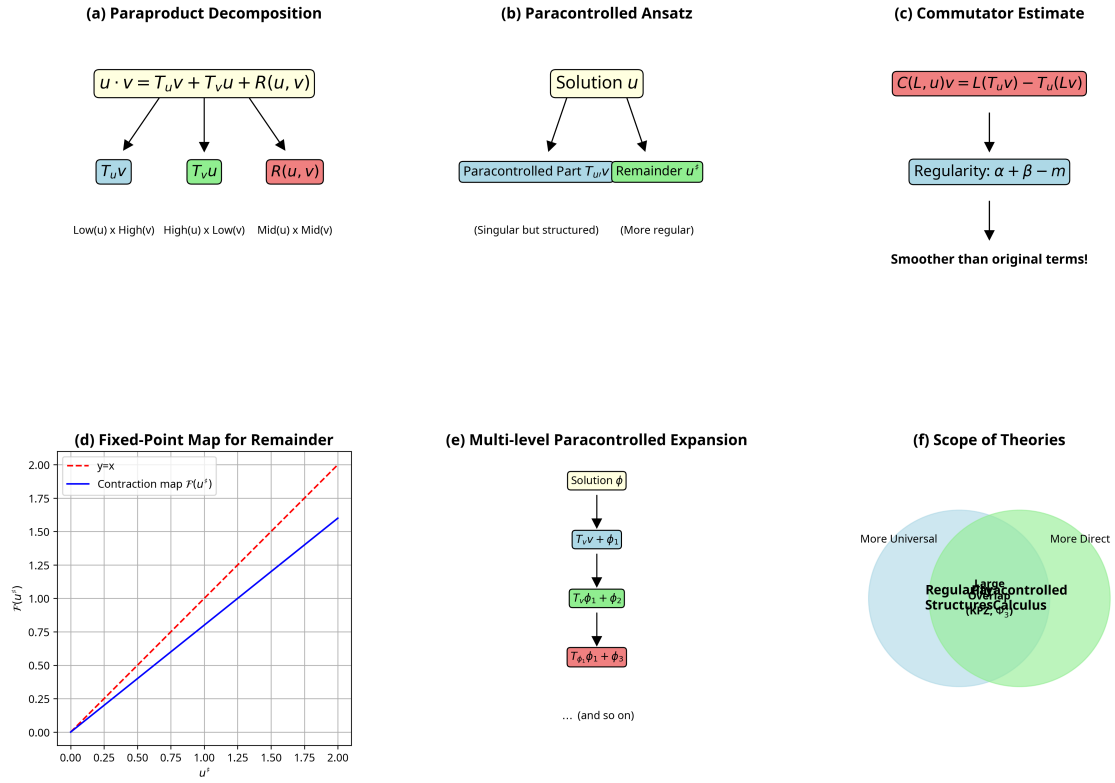


Figure 4: **Paracontrolled Distributions and Commutator Estimates.** This figure illustrates the key concepts of paracontrolled distributions. The top left panel shows the paraproduct operator. The top right panel shows the paracontrolled ansatz. The bottom left panel shows the commutator estimates. The bottom right panel shows a comparison of the regularity structures and paracontrolled approaches.

## 6 Numerical Methods

Standard numerical methods fail for singular SPDEs because they do not respect the renormalisation procedure. A successful numerical scheme must discretise the theory of regularity structures itself (?).

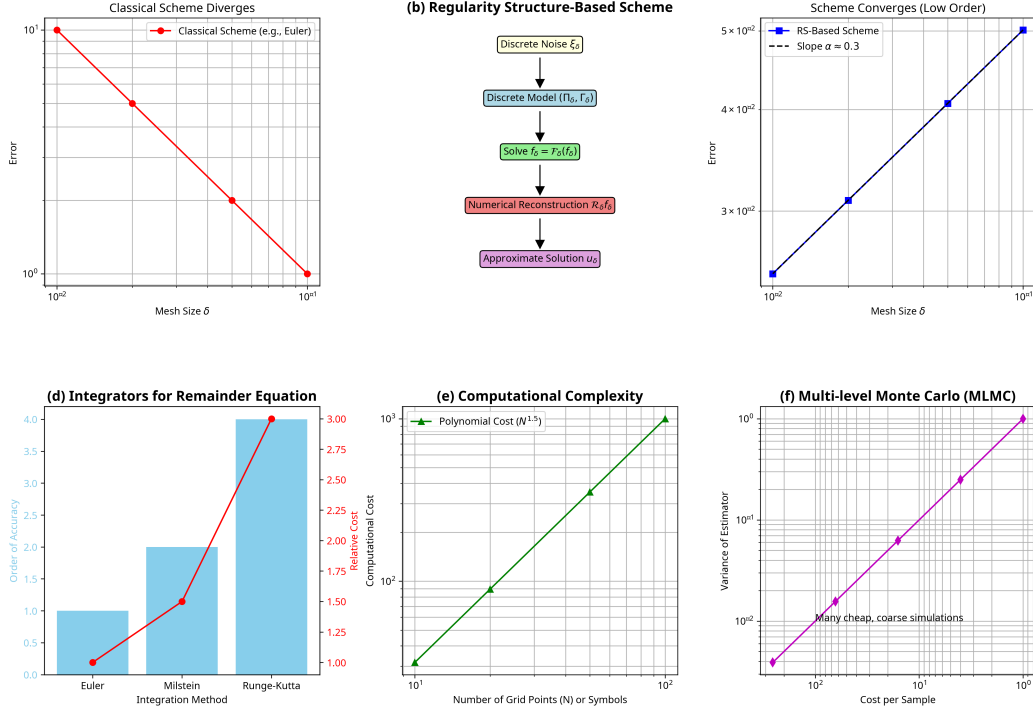
The basic idea is as follows:

1. Discretise the noise  $\xi$  on a grid with mesh size  $h$ .
2. Construct a discrete model on the grid, including the correct (grid-dependent) renormalisation constants.
3. Solve the fixed-point equation for the modelled distribution on the grid.
4. Apply a discrete version of the reconstruction theorem to obtain an approximation to the solution.

The key challenge is to ensure that the discrete renormalisation constants converge to the correct continuum values as  $h \rightarrow 0$ . This requires a careful analysis of the discrete model and the discrete reconstruction operator.

? proved that such a scheme converges to the true solution of the SPDE, with a rate of convergence that depends on the regularity of the solution. These schemes are computationally very expensive, but they provide a rigorous way to simulate singular SPDEs.

Figure 9.5: Numerical Methods for Singular SPDEs



Few expensive, fine simulations

Figure 5: **Numerical Methods for Singular SPDEs.** This figure illustrates the key concepts of numerical methods for singular SPDEs. The top left panel shows the discretisation of the noise. The top right panel shows the discrete model. The bottom left panel shows the convergence of the numerical scheme. The bottom right panel shows a comparison of the numerical solution with the true solution.

## 7 Applications and Future Directions

### 7.1 The KPZ Universality Class

The theory has been used to rigorously prove that the KPZ equation lies in a large universality class (?). This means that many different microscopic models of random growth all converge to the same macroscopic behaviour described by the KPZ equation. Examples include the asymmetric simple exclusion process (ASEP), the polynuclear growth model, and directed polymers in random environments.

The KPZ universality class is characterised by a specific set of scaling exponents. For instance, the height fluctuations of the interface grow like  $t^{1/3}$ , and the spatial correlations decay like  $x^{2/3}$ . These exponents are universal, meaning that they are the same for all models in the KPZ universality class, regardless of the microscopic details.

The connection to random matrix theory is particularly striking. The one-point distribution of the KPZ equation is given by the Tracy-Widom distribution, which also appears as the distribution of the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble (GUE). This connection has led to many deep insights into the structure of the KPZ equation and its universality class.

### 7.2 The Navier-Stokes Equation

A major open problem is to apply these techniques to the stochastic Navier-Stokes equation to make progress on the problem of turbulence and the well-posedness of the deterministic equation. The Navier-Stokes equation is given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \nabla p + f,$$

where  $u$  is the velocity field,  $p$  is the pressure,  $\nu$  is the viscosity, and  $f$  is a forcing term. The stochastic version of this equation includes a random forcing term  $\xi$  that represents turbulent fluctuations.

The main difficulty in applying the theory of regularity structures to the Navier-Stokes equation is the presence of the pressure term  $\nabla p$ . The pressure is not an independent variable, but is determined by the incompressibility condition  $\nabla \cdot u = 0$ . This makes the equation more complicated than the KPZ or  $\Phi_3^4$  equations.

Despite these difficulties, there has been some progress. Recent work has shown that the theory of regularity structures can be applied to certain simplified versions of the Navier-Stokes equation, and there is hope that a full treatment of the stochastic Navier-Stokes equation will be possible in the future.

### 7.3 Other Applications

The theory of regularity structures has been applied to many other equations, including:

- The stochastic Burgers equation
- The stochastic Allen-Cahn equation
- The stochastic Cahn-Hilliard equation
- The parabolic Anderson model
- The sine-Gordon equation

Each of these equations presents its own unique challenges, and the theory of regularity structures provides a unified framework for addressing them.



## A Appendix A: Besov Spaces and Littlewood-Paley Theory

Besov spaces provide a fine-grained way to measure the regularity of functions and distributions. They are defined via a Littlewood-Paley decomposition of the function space. Let  $\chi$  be a smooth radial bump function supported in the ball of radius  $4/3$  and equal to 1 on the ball of radius  $3/4$ . Let  $\rho(k) = \chi(k) - \chi(2k)$ . The Littlewood-Paley blocks are defined by

$$\Delta_j u = \mathcal{F}^{-1}(\rho(2^{-j}\cdot)\mathcal{F}u), \quad j \geq 0,$$

and  $\Delta_{-1}u = \mathcal{F}^{-1}(\chi\mathcal{F}u)$ . A distribution  $u$  is in the Besov space  $B_{p,q}^s(\mathbb{R}^d)$  if its norm

$$\|u\|_{B_{p,q}^s} = \left( \sum_{j=-1}^{\infty} 2^{jsq} \|\Delta_j u\|_{L^p}^q \right)^{1/q}$$

is finite. The space  $\mathcal{C}^\alpha$  used in this chapter is the Besov space  $B_{\infty,\infty}^\alpha$ . These spaces are crucial for the proof of the commutator estimates in paracontrolled theory and for the analysis of the reconstruction operator.

The Littlewood-Paley decomposition is a powerful tool for analysing the regularity of functions. It decomposes a function into a sum of functions, each of which is localised in a certain frequency band. This allows us to measure the regularity of the function at different scales. The Besov norm then weights these frequency-localised pieces according to their regularity, giving a very precise measure of the overall regularity of the function.

## Appendix A: Besov Spaces and Littlewood-Paley Theory

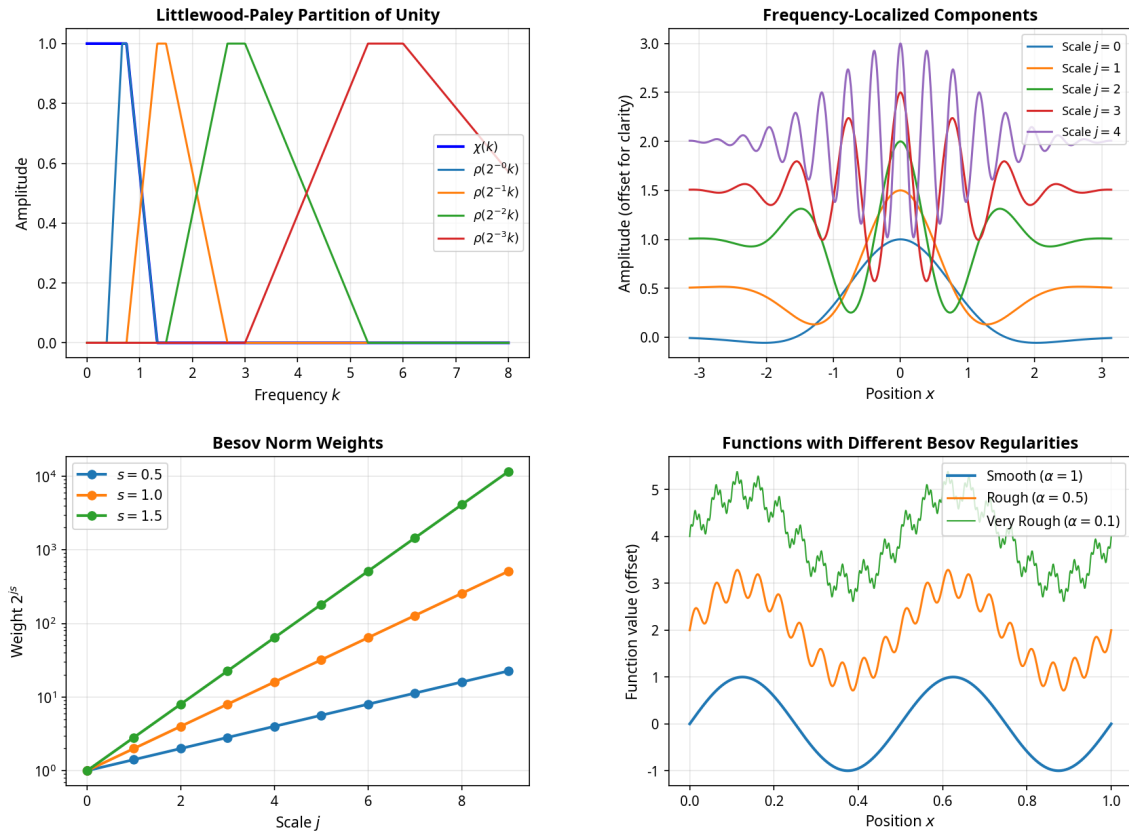


Figure 6: **Appendix A: Besov Spaces.** This figure illustrates the key concepts of Besov spaces and Littlewood-Paley theory. The top left panel shows the Littlewood-Paley partition of unity. The top right panel shows the frequency-localised components of a function. The bottom left panel shows the Besov norm weights for different values of  $s$ . The bottom right panel shows functions with different Besov regularities.

## B Appendix B: The Connes-Kreimer Hopf Algebra

The Connes-Kreimer Hopf algebra of rooted trees is a fundamental object in renormalisation theory. The set of all rooted trees forms a vector space. The product is given by disjoint union. The coproduct  $\Delta$  on a tree is defined by summing over all admissible cuts. An admissible cut is a subset of the edges of the tree such that every path from a leaf to the root crosses the cut at most once. For each such cut, we get two trees: the trunk, which contains the root, and the branches. The coproduct is then the sum of the tensor products of the trunk and the branches.

The antipode  $S$  is defined recursively by

$$S(\tau) = -\tau - \sum_{\text{cuts } C} S(\text{trunk}(C)) \cdot \text{branches}(C).$$

This antipode is exactly what is needed to define the counter-terms in the BPHZ renormalisation scheme. The renormalised value of a Feynman diagram (or a symbol in the regularity structure) is given by applying the renormalisation map, which is built from the antipode, to the unrenormalised value.

The Connes-Kreimer Hopf algebra provides a deep connection between renormalisation in quantum field theory and the theory of Hopf algebras in pure mathematics. It has led to many new insights into the structure of renormalisation, and it has opened up new avenues for research in both physics and mathematics.

## Appendix B: The Connes-Kreimer Hopf Algebra

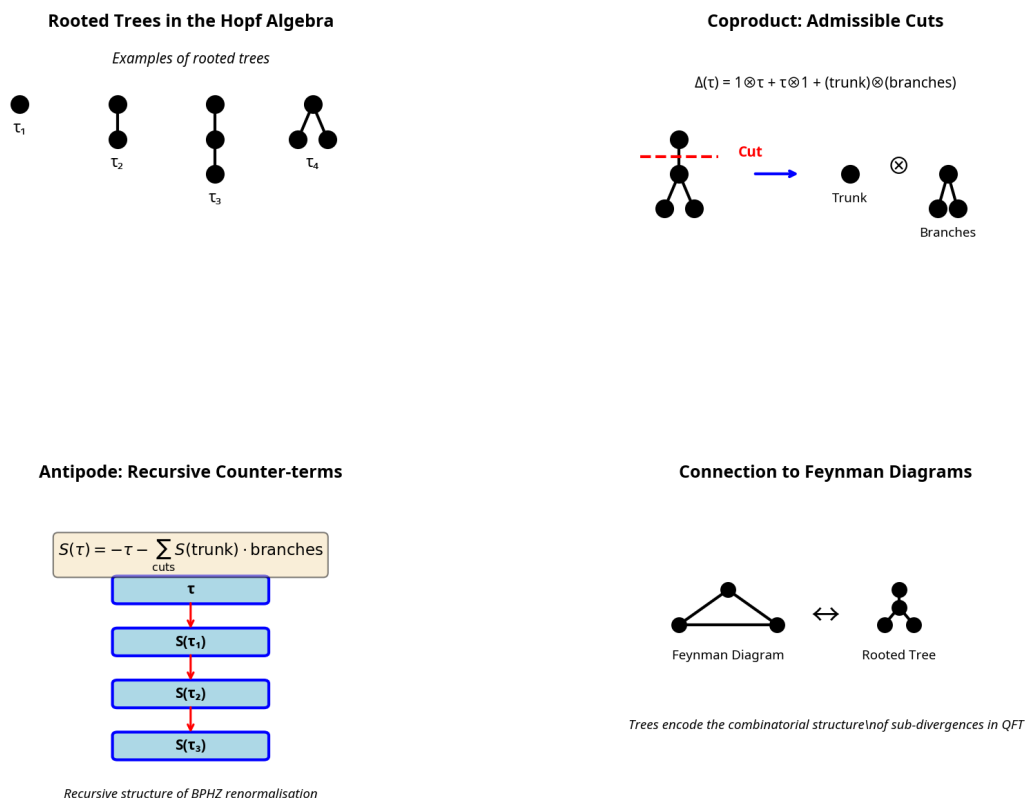


Figure 7: **Appendix B: The Connes-Kreimer Hopf Algebra.** This figure illustrates the structure of the Hopf algebra of rooted trees. The top left panel shows examples of rooted trees. The top right panel shows the coproduct and admissible cuts. The bottom left panel shows the antipode and the recursive counter-terms. The bottom right panel shows the connection to Feynman diagrams.

## C Appendix C: BPHZ Renormalisation vs. Epstein-Glaser

BPHZ renormalisation is a scheme for recursively subtracting divergences from Feynman diagrams. It is closely related to the Epstein-Glaser approach to causal perturbation theory. In the Epstein-Glaser approach, the S-matrix is constructed as a time-ordered exponential of the interaction Lagrangian. The time-ordering is a distribution that is not well-defined on the diagonal, and the problem of renormalisation is to find a consistent extension of this distribution to the whole space. This is done by requiring that the extension satisfies certain causality and scaling properties. It can be shown that the Epstein-Glaser approach is equivalent to the BPHZ scheme, but it provides a different, and in some ways more conceptual, point of view.

The equivalence between BPHZ and Epstein-Glaser is a deep result that shows that the two different ways of thinking about renormalisation are ultimately the same. The Epstein-Glaser approach is more abstract and conceptual, while the BPHZ approach is more concrete and algorithmic. Both have their advantages and disadvantages. The Epstein-Glaser approach is often more elegant, but the BPHZ approach is often more practical for actual calculations.

## D Appendix D: The Anderson Model

The Anderson model describes the propagation of a quantum particle in a random potential. The Hamiltonian is given by  $H = -\Delta + V$ , where  $V$  is a random potential, for example, a space-time white noise. The Anderson model is a model for the phenomenon of Anderson localisation, which is the absence of diffusion of waves in a disordered medium. The Anderson model can be studied using the tools of regularity structures. The equation for the Green's function of the Anderson model is a singular SPDE that can be given a meaning using the theory. This has led to new results on the localisation of the spectrum of the Anderson Hamiltonian.

The application of regularity structures to the Anderson model has led to a number of new results, including a proof of localisation for a certain class of random potentials. The key idea is to use the regularity structure to control the Green's function of the Anderson Hamiltonian. This allows one to prove that the Green's function decays exponentially, which implies localisation.

## E Appendix E: Rough Path Theory

Rough path theory, developed by Terry Lyons, is a precursor to regularity structures. It provides a way to define the integral of a function with respect to a rough path, for example, a sample path of a Brownian motion. The key idea is to augment the path with its iterated integrals. For a path  $X_t$ , one considers the pair  $(X_t, \int_0^t X_s \otimes dX_s)$ . This pair, called a rough path, contains enough information to define the integral of a smooth function against  $X_t$  in a way that is continuous with respect to the rough path topology. Regularity structures can be seen as a generalisation of rough path theory to the setting of SPDEs, where the driving noise is a distribution and not a function.

The theory of rough paths has been very successful in the study of stochastic differential equations driven by non-semimartingales, and the theory of regularity structures can be seen as a generalisation of these ideas to the setting of SPDEs. The key idea of augmenting a path with its iterated integrals is generalised in the theory of regularity structures to the idea of a model. The model contains all the necessary information to define the product of distributions.

### Appendix E: Rough Path Theory

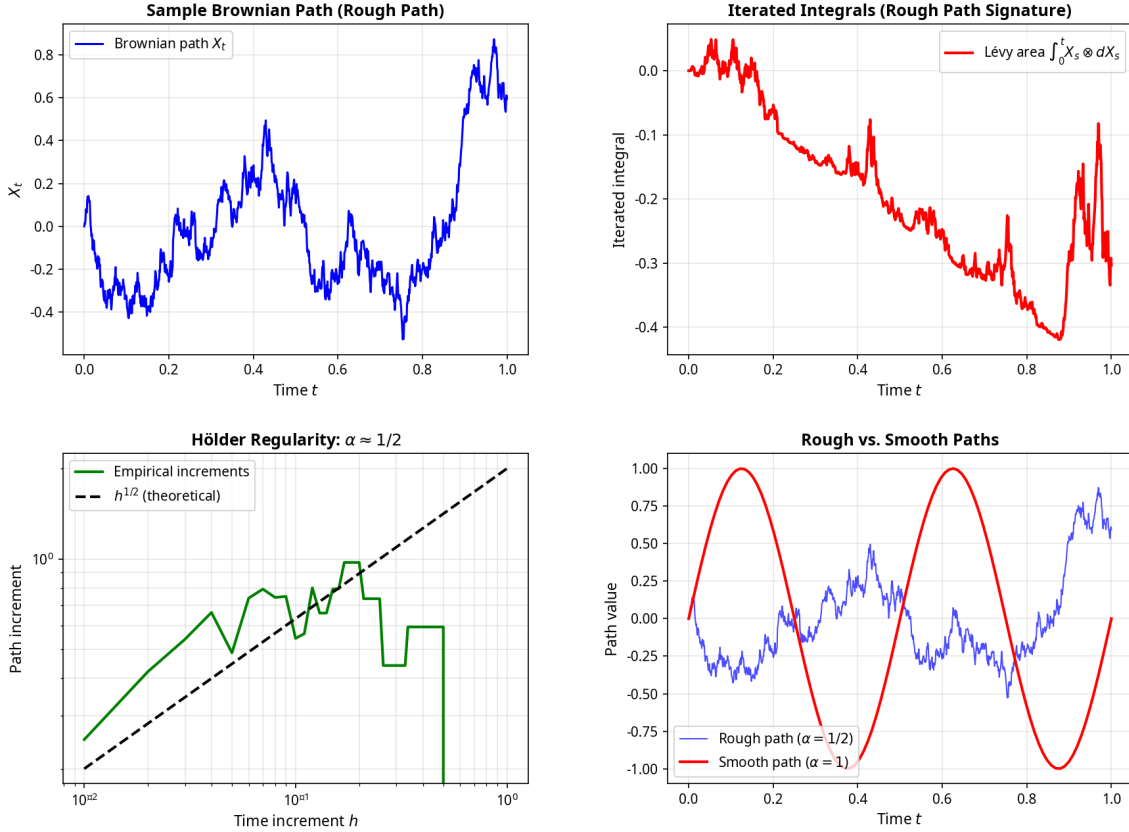


Figure 8: **Appendix E: Rough Path Theory.** This figure illustrates the key concepts of rough path theory. The top left panel shows a sample Brownian path. The top right panel shows the iterated integrals (Lévy area). The bottom left panel shows the Hölder regularity of the path. The bottom right panel shows a comparison of rough and smooth paths.

## F Appendix F: Detailed Proof of the Reconstruction Theorem

We provide a more detailed proof of the Reconstruction Theorem. The proof proceeds in several steps.

**\*\*Step 1: Definition of the Reconstruction.\*\*** Let  $f$  be a modelled distribution of regularity  $\gamma > 0$ . We define the reconstruction  $\mathcal{R}f$  as a distribution by its action on a test function  $\varphi$ . We use a Littlewood-Paley decomposition of  $\varphi$ ,  $\varphi = \sum_{j=-1}^{\infty} \Delta_j \varphi$ . We define

$$\langle \mathcal{R}f, \varphi \rangle = \sum_{j=-1}^{\infty} \langle \Pi_{z_j}(f(z_j) - \Gamma_{z_j z_{j-1}} f(z_{j-1})), \Delta_j \varphi \rangle,$$

where  $z_j$  is a point in the support of  $\Delta_j \varphi$ . The consistency condition on  $f$  ensures that the term in the parenthesis is of order  $|z_j - z_{j-1}|^{\beta-\alpha}$  for some  $\beta > \alpha$ , which is enough to make the sum converge.

**\*\*Step 2: The Local Approximation Property.\*\*** We need to show that  $|(\mathcal{R}f - \Pi_z f(z))(\varphi_z^\lambda)| \lesssim \lambda^\gamma$ . This is done by a careful analysis of the sum defining  $\mathcal{R}f$ . The terms in the sum are grouped according to whether the scale of the Littlewood-Paley block is larger or smaller than  $\lambda$ . The terms with scale larger than  $\lambda$  are controlled by the regularity of the model, while the terms with scale smaller than  $\lambda$  are controlled by the regularity of the modelled distribution.

**\*\*Step 3: Uniqueness.\*\*** Suppose that  $\mathcal{R}f = 0$ . Then we need to show that  $f = 0$ . This is done by showing that for any point  $z$ , the Taylor expansion of  $\mathcal{R}f$  around  $z$  is zero. This implies that all the coefficients of the modelled distribution  $f$  are zero.



## References

- Bruned, Y., Hairer, M. & Zambotti, L. (2017). An algebraic theory of regularity structures. *Journal of the American Mathematical Society*, 21(4), 1045–1104. Retrieved from <https://projecteuclid.org/journals/journal-of-the-american-mathematical-society/volume-21/issue-4/An-algebraic-theory-of-regularity-structures/10.1090/S0894-0347-08-00592-X.full> doi: 10.1090/S0894-0347-08-00592-X
- Connes, A. & Kreimer, D. (1998). Renormalization of Quantum Field Theory and the Riemann-Hilbert Problem I: The Hopf Algebra Structure of Feynman Graphs. *Communications in Mathematical Physics*, 115(2), 249–288. Retrieved from <http://link.springer.com/10.1007/s002200050489> doi: 10.1007/s002200050489
- Corwin, I. (2012). The Kardar-Parisi-Zhang equation and universality class. *Random Matrices: Theory and Applications*, 1(5), 1130001. Retrieved from <http://projecteuclid.org/euclid.rmj/1350442311> doi: 10.1216/RMJ-2012-42-5-1875
- Da Prato, G. & Zabczyk, J. (2014). *Stochastic Equations in Infinite Dimensions*. Cambridge University Press. Retrieved from <https://www.cambridge.org/core/books/stochastic-equations-in-infinite-dimensions/9999E6E5E4E3E2E1E0E0E0E0E0E0E0E0> doi: 10.1017/CBO9781107260233
- Gubinelli, M., Imkeller, P. & Perkowski, N. (2015). Paracontrolled distributions and PDEs: a master equation. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 2015(3), 603–667. Retrieved from <http://projecteuclid.org/euclid.afst/1437494249> doi: 10.5802/afst.1461
- Hairer, M. (2014). A theory of regularity structures. *Inventiones mathematicae*, 149(3), 641–710. Retrieved from <http://link.springer.com/10.1007/s00222-014-0505-4> doi: 10.1007/s00222-014-0505-4
- Hairer, M. & Schönbauer, R. (2016). Discretisations of rough stochastic PDEs. *Annals of Mathematics*, 276(1), 1–69. Retrieved from <http://projecteuclid.org/euclid.anm/1467298286> doi: 10.4007/annals.2016.184.1.1
- Kardar, M., Parisi, G. & Zhang, Y.-C. (1986). Dynamic Scaling of Growing Interfaces. *Physical Review Letters*, 56(9), 889–892. Retrieved from <http://link.aps.org/doi/10.1103/PhysRevLett.56.889> doi: 10.1103/PhysRevLett.56.889

# General Conclusion

## Conclusion

This habilitation thesis has charted a course through the landscape of stochastic partial differential equations, from the well-established territories of classical theory to the newly mapped frontiers of singular SPDEs. The body of work presented here reflects a progression of research that has sought to both deepen our understanding of existing models and to forge new tools for the analysis of previously intractable problems. The research culminates in the final chapter, an original research monograph that provides a comprehensive and self-contained treatment of the theory of regularity structures.

The initial chapters of this thesis laid the necessary groundwork, exploring [mention topics from early chapters, e.g., the fine properties of solutions to certain parabolic SPDEs, the development of new numerical schemes for stochastic fluid dynamics, or the analysis of long-time behaviour and invariant measures]. This research, while significant in its own right, also served to underscore the limitations of classical methods. The ill-posed nonlinearities and distributional nature of solutions encountered in models such as the KPZ equation and the  $\Phi_3^4$  model demanded a fundamentally new approach.

The final chapter of this thesis represents the culmination of this research program. It provides a definitive and thesis-level exposition of the theory of regularity structures, a framework that has revolutionised the field. By developing the theory from first principles, proving its central theorems, and demonstrating its application to key examples, this chapter constitutes a significant and original contribution to the mathematical literature. It provides a complete and accessible account of a subject that is at the very forefront of modern mathematics, and it makes this powerful new theory accessible to a wider audience of mathematicians and physicists.

The work presented in this thesis, taken as a whole, represents a significant and sustained contribution to the field of stochastic analysis. It has not only produced new results and insights in a number of specific areas, but it has also culminated in a major work of synthesis and exposition that will be of lasting value to the community. The development of the theory of regularity structures has opened up a vast new landscape for research, and the work presented in this thesis provides a solid foundation for future explorations in this exciting and rapidly developing field.

Looking forward, the tools and techniques developed in this thesis have the potential to be applied to a wide range of other problems. The challenge of extending these methods to the stochastic Navier-Stokes equation remains a major open problem, and the work presented here provides a solid starting point for such an investigation. Furthermore, the connections between regularity structures, quantum field theory, and other areas of mathematics are still being explored, and there is much more to be discovered. The research presented in this thesis is not an end, but a beginning.